# Remember the Past and Forget Thresholds

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## Abstract

One of the most popular existing models for task allocation in ant colonies is the so-called *threshold-based* task allocation model. Here, each ant has a fixed, and possibly distinct, threshold. Each task has a fixed *demand* which represents the number of ants required to perform the task.[1] The *stimulus* an ant receives for a task is defined as the demand of the task minus the number of ants currently working at the task. An ant joins a task if the stimulus of the task exceeds the ant's threshold.

A large body of results has studied this model for over four decades; however, most of the theoretical works focuses on the study of two tasks. Interestingly, no work in this line of research shows that the number of ants working at a task eventually converges towards the demand nor does any work bound the distance to the demands over time.

In this work, we study precisely this convergence. Our results show that while the threshold-based model works fine in the case of two tasks (for certain distributions of thresholds); the threshold model no longer works for the case of more than two tasks. In fact, we show that there is no possible setting of thresholds that yields a satisfactory deficit (demand minus number of ants working on the task) for each task.

This is in stark contrast to other theoretical results in the same setting [CDLN14] that rely on state-machines, i.e., some form of small memory together with probabilistic decisions. Note that, the classical threshold model assumes no states or memory (apart from the bare minimum number of states required to encode which task an ant is working on). The resulting task allocation is near-optimal and much better than what is possible using joining thresholds. This remains true even in a noisy environment [DLM+18].

While the deficit is not the only important metric, it is conceivably one of the most important metrics to guarantee the survival of a colony: for example if the number of workers assigned for foraging stays significantly below the demand, then starvation may occur. Moreover, our results do not imply that ants do not use thresholds; we merely argue that relying on thresholds yields a considerable worse performance.

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[1] Some models also consider demands that increase over time.

## 1 Introduction

The theoretical study of threshold-based division of labor in insect colonies dates back to the work of Wilson [Wil76]. A particular species of ants (*P. dentata*) was observed to split into two castes (majors and minors) where each caste performs a different set of tasks.

The authors of [BTD96] built on the observations of Wilson [Wil76] and suggested a theoretical model of fixed thresholds to explain task allocation. The assumption of this model is that each ant is characterized by a fixed (genetically determined) response threshold to various stimuli. The authors assumed a single task and two castes of ants with respective thresholds $\theta_1, \theta_2$. They did not assume a fixed demand for each task, but rather, a total number of units of work that must be done by the ants in each step. If work is not being done, the units of work accumulate and must be handled later.

The authors focused on the average time spent working as a function of the fraction of ants of caste $i$ in the colony. They showed that their simulation results coincide with empirical results of ants *P. megacephala* and the results observed by [Wil76] and therefore account for task specialization. However, they did not provided a quality measure for task allocation. Moreover, our simulations of their algorithm indicate that the system does not converge to such an equilibrium and instead results in very large oscillations (which are linear in the number of workers). Bonabeau et al. [BTD98] extended the theory of fixed response thresholds in various ways. They showed that this model is flexible enough to explain many phenomena observed in the experiments, like temporal polyethism and task specialization. The paper also contains simulations regarding temporal polyethism.

A model of thresholds proposed by Page *et al.* [PM98] differs from the line of work by Bonabeau *et al.* in the following aspects. The model considered by Page *et al.* assumes purely deterministic task allocation (*i.e.,* the decision by each ant is made deterministically based only on the level of stimulus and the threshold of this ant). The stimulus does not accumulate over time. The level of threshold of each ant is assumed to be fixed and initially drawn from different distributions. However, the desired demand of the task is not defined, hence, the authors do not analyze any quality measure for the allocation. The evolution of thresholds over time or generations was also studied in later works [DPKW12; LTKF12].

[MWCG16] considered a modification of the response threshold model from [BTD96]. The authors first present results of an experiment on bumblebees with one task that concern the maintenance of a correct temperature of the brood. The authors argue that the standard probabilistic threshold model of [BTD96] does not match the experimental results because in the experiment the concentration of the stimulus did not significantly influence the rate of engagement of worker with the task. In a second step, a new time-resolved model (which can be seen as a simple state machine) is presented. State-machines have also been used by [PGG96], where the authors studied the influence of the group size on task allocation in a very general theoretical model of interacting social insects.

**Distributed Computing Background.** Task allocation has recently gained attention from theoretical computer scientists. In [CDLN14], the authors assumed synchronous rounds and binary feedback. Each task has an associated demand and whenever too many ants work on a task, all ants receive the feedback `overload` and `lack` otherwise. It turns out that this feedback is enough to allow the ants to converge to an almost-optimal allocation. It is worth emphasizing that in [CDLN14] no thresholds are used. Considering a noisy version of the model was left as an open question, which was later studied by [DLM+18]. In their setting, the binary feedback is correct with a probability that depends on how large the absolute value of the deficit is; the further the deficit is from 0, the more likely that the ants receive the correct feedback. The authors use a sigmoid function of the deficit to model this. They showed that convergence to an almost perfect deficit cannot be achieved; small oscillations are unavoidable. However, they also presented a simple algorithm that convergences to states with small oscillations. Both papers use as a quality measure the absolute deficit of the tasks and show that the number of ants working on a task quickly approaches the demand. Both extend to the setting of dynamic task allocation, where the demands are changed from time to time.

## 2 Formal Model, Notation and Assumptions

We assume we have $n$ ants and $k$ tasks, where $k$ is some constant. Furthermore, each task $j \in [k] = \{1, 2, \ldots, k\}$ has a fixed *demand* $d^{(j)}$ meaning that the task requires $d^{(j)}$ many ants assigned to it, *e.g.*, if task $j$ represents nursing for brood, then $d^{(j)}$ represents the required number of nursing ants. Let $\mathbf{d} = [d^{(1)}, d^{(2)}, \ldots, d^{(k)}]$ be the demand vector. We assume that $\sum_{j \in [k]} d^{(j)} \leq n$. Let $W_t^{(j)}, j \in [k]$ denote the *load* of resource $j$ at time $t$, *i.e.*, the number of ants performing the task. For task $j \in [k]$ we define the *deficit* as $\Delta_t^{(j)} = d^{(j)} - W_t^{(j)}$: a positive deficit signifies that ants are missing and a negative deficit signifies an overload of ants. We assume that each ant $i \in [n]$ has for each task $j \in [k]$ a *joining threshold* $j_i^{(j)}$ and *leaving threshold* $\ell_i^{(j)} \geq 0$. We assume synchronous rounds, during which, the thresholds are used by the ants to make (simultaneous) decisions about joining or leaving the tasks. When ant $i$ is working on task $j$ with overload of at least $\ell_i^{(j)}$, then ant $i$ leaves the task. Symmetrically, if there is exactly one task $j$ with underload of at least $j_i^{(j)}$, then ant $i$ joins the task; if there are several such tasks, ant $i$ chooses one uniformly at random. More formally, if ant $i$ is working on task $j$ in some fixed round $t$ and $-\Delta_{t-1}^{(j)} \geq \ell_i^{(j)}$, then $\texttt{task}_i(t) = \text{idle}$, else $\texttt{task}_i(t) = \texttt{task}_i(t-1)$. If $i$ is idle ($\texttt{task}_i(t-1) = \text{idle}$), then $\texttt{task}_i(t)$ is drawn uniformly from $\{j \colon \Delta_{t-1}^{(j)} \geq j_i^{(j)}\}$; if the set is empty, then the ant remains idle. We assume there is no communication among the ants. The question we ask here is whether there exists a set of thresholds such that for any set of demands, the deficit per task is eventually sub-linear in the number of ants.

## 3 Results and Proofs

For two tasks, the following simple assignment of joining thresholds $j_i^{(1)} = i, j_i^{(2)} = n - i + 1$, for $i \in [n]$ and equal leaving thresholds $\ell_i^{(1)} = \ell_i^{(2)} = 1$ achieves a perfect allocation after one step. This is summarized in the following generalization of [PM98].

**Proposition 3.1.** *For $k \leq 2$, the above assignment of joining and leaving thresholds (all leaving thresholds equal), ensures that for any demand vector $\mathbf{d}$ the resulting allocation of ants to tasks yields deficit equal to $0$ in all the tasks.*

However for $k > 2$ such an assignment of thresholds is no longer possible as we show in Theorem 3.2. The negative result still holds even if leaving thresholds instead of joining thresholds are used (Theorem 3.3). The idea behind the impossibility results is combinatorial: regardless of how the thresholds are chosen, there is always a distribution of demands that will result in a large deficit. The challenge lies in showing that this remains true over arbitrarily large spans of time.

**Theorem 3.2** (Only Joining Thresholds). *Assume that $k \geq 3$ and for any task $j \in [k]$ ants have arbitrary joining thresholds $j_1^{(j)}, j_2^{(j)}, \ldots, j_n^{(j)}$ and equal leaving thresholds, i.e., $\ell_i^{(j)} = \ell_{i'}^{(j')}$ for all $i, i' \in [n]$ and $j, j' \in [k]$. Then there exists a demand vector such that there is task $j \in [k]$ having an absolute deficit of $\Omega(n)$ in every round.*

*Proof.* Consider any colony of $n$ ants with arbitrary joining thresholds and identical leaving thresholds. We define sets: $A_j = \left\{ i \colon j_i^{(j)} \leq \frac{9n}{20} \right\}$. First we want to show that for each $j \in [k]$ we have:

$$n \cdot \left( \frac{9}{20} + \frac{1}{50k} \right) \geq |A_j| \geq n \cdot \left( \frac{2}{5} - \frac{1}{50k} \right), \tag{1}$$

because if this double inequality is not satisfied for some $j^* \in [k]$, then the adversary can set the demand vector to be $d^{(j^*)} = 2n/5$ and $d^{(j)} = 0$ for $j \neq j^*$. In this case, the absolute value of the deficit is $\Omega(n/k)$ in each step $t > 1$. Thus, in the following we assume (1) holds for all $j \in [k]$. We have by inclusion-exclusion principle:

$$n \geq \left| \bigcup_{i \in [k]} A_i \right| \geq \sum_{i \in [k]} |A_i| - \sum_{1 \leq i < j \leq k} |A_i \cap A_j|,$$

and

$$\sum_{1\le i<j\le k} |A_i \cap A_j| \ge \sum_{i\in[k]} |A_i| - n \ge n\left(\frac{45k-2}{100} - 1\right).$$

The sum on the left side has $\binom{k}{2}$ summands thus we can find two tasks $i^*, j^* \in [k]$ such that:

$$|A_{i^*} \cap A_{j^*}| \ge \frac{n}{\binom{k}{2}} \cdot \left(\frac{45k-102}{100}\right) = \frac{n}{k}\left(\frac{9}{10} - \frac{57}{50(k-1)}\right). \tag{2}$$

The adversary sets the demand vector to be $d^{(i^*)} = d^{(j^*)} = 9n/20$ and $d^{(j)} = 0$, for $j \in [k] \setminus \{i^*, j^*\}$. The set of ants that respond to such demands is exactly $A_{i^*} \cup A_{j^*}$. We have by (2):

$$\begin{aligned}|A_{i^*} \cup A_{j^*}| = |A_{i^*}| + |A_{j^*}| - |A_{i^*} \cap A_{j^*}| &\le n \cdot \left(\frac{9}{10} + \frac{1}{25k}\right) - \frac{n}{k}\left(\frac{9}{10} - \frac{57}{50(k-1)}\right)\\ &= n \cdot \left(\frac{9}{10} - \frac{3}{5k} + \frac{57}{50k(k-1)}\right) = \frac{9n}{10} - \frac{n}{k}\left(\frac{43}{50} - \frac{57}{50(k-1)}\right).\end{aligned}$$

Note that for $k \ge 3$, value $\left(\frac{43}{50} - \frac{57}{50(k-1)}\right)$ is a positive constant. Notice that in any step $t$ only ants from $A_{i^*} \cup A_{j^*}$ can possibly be working on tasks $i^*, j^*$ (because all other ants have a too high threshold to join). Moreover we showed that the total size of this set is too small to satisfy the demands of both tasks. It does not matter how the ants from $A_{i^*} \cup A_{j^*}$ distribute themselves among these two tasks there will be at least $\Omega(n/k) = \Omega(n)$ deficit and this deficit will occur in every step. ☐

We now give the lower bound for the case when only leaving thresholds are used.

**Theorem 3.3** (Only Leaving Thresholds). *Assume that $k \ge 3$ and $n$ ants have arbitrary leaving thresholds $\ell_1^{(j)}, \ell_2^{(j)}, \dots, j_n^{(j)}$ and equal joining thresholds, i.e., $j_i^{(j)} = j_{i'}^{(j')}$ for all $i, i' \in [n]$ and $j, j' \in [k]$. Then, for $n$ large enough, there exists a demand vector such that there is a task with a deficit of $\Omega(n)$ in every round with overwhelming probability $1 - \exp(-\Omega(n))$.*

*Proof.* We start by establishing two claims (Claim 3.4 and Claim 3.5). Each claim states that if a certain property of the thresholds fails, then the deficit must be $\Omega(n)$. Due to space imitation, the proofs of the claims are omitted.

Let $A_j$ be the set of ants that leave task $j$ if the overload is less than $\frac{n}{5}$, *i.e.*, $A_j = \left\{i \colon \ell_i^{(j)} \le \frac{n}{5}\right\}$.

**Claim 3.4.** *Let $\varepsilon = 1/30$. If the joining threshold is smaller than $(1-\varepsilon)n$, then the deficit is $\Omega(n)$.*

**Claim 3.5.** *Let $\varepsilon = 1/30$. If $|A_j| \notin [n/3 - \varepsilon n, n/3 + \varepsilon n]$, then the deficit is $\Omega(n)$.*

Let $\varepsilon = 1/30$. Due to Claim 3.4 and Claim 3.5, we can focus on the case that $|A_j| \in [n/3 - \varepsilon n, n/3 + \varepsilon n]$ and $j_1^{(1)} \ge (1-\varepsilon)n$. Recall that we assume that if there are several tasks with a positive deficit (lack) of ants, then each ant selects one of these uniformly at random. In the remainder, consider the setting where $d^{(1)} = d^{(2)} = n/9$, $d^{(3)} = 7n/9$, and for $j \in \{4, \dots, k\}$ the demands are set $d^{(j)} = 0$.[2] By Chernoff bounds, we have for $j \in [3]$ that $W_1^{(j)} = (1 \pm \varepsilon)n/3$. Note that task 3 is underloaded and thus, by assumption, no ants will leave this task. Furthermore for $j \in [2]$ the overload will be

$$W_1^{(j)} - d^{(j)} \in [(1-\varepsilon)(n/3) - n/9, (1+\varepsilon)(n/3) - n/9].$$

Observe that the overload is upper bounded by $n/3$. The number of ants of $A_j$ is at most $(1 \pm 2\varepsilon)n/9$ for each $j \in [3]$ with probability at least $1 - \exp(-\Omega(n))$. Hence, the total number of ants per task $j \in [2]$ after leaving (which is limited to the ants in $A_j$) is at least $(1-\varepsilon)(n/k) - (1+2\varepsilon)n/9 > n/9 = d^{(j)}$ and hence the task remains overload even after ants leave the task.

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[2] It is easy to extend the proof to nonzero demands.

In the next round, all idle ants will join task $k$ or remain idle; since task $k$ is the only task with a deficit. The number of idle ants in is bounded by $(1+2\varepsilon)n/3+\varepsilon n$ and the load of task $k$ becomes

$$W_3^{(k)} \leq (1+\varepsilon)n/3 + (1+2\varepsilon)n/3 + \varepsilon n \leq 6.9/9,$$

assuming $k \geq 3$. This corresponds to a deficit of $\Delta_3^{(1)} \geq 7n/9 - 6.9n/9 = \Omega(n)$. The load of all other tasks remains unchanged and the load of task $k$ will also not change afterwards. Hence, the deficit is $\Omega(n)$ in every round from here on, which completes the proof. $\square$

**Future Work.** It remains an interesting open question to study the case where both joining and leaving thresholds are used by the ants. So far, leaving thresholds have not been considered in the biology literature. Nonetheless, we conjecture that in the model without memory, even if both joining and leaving thresholds are used, convergence to any distribution of demands cannot be guaranteed. In addition, our work focuses on the worst-case by showing that for any distribution of thresholds there exists a distribution of demands such that the deficit is large. However, we conjecture that the results also hold for almost any distribution of demands; however, quantifying the deficit over large periods of time appears to be intractable. To circumvent this, it might be interesting to study the case where there are only a small number of different thresholds. Most importantly, we ask whether experiments on ants could suggest that the ants' decisions can be explained by state-machines rather than response thresholds.

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## References

[BTD96] E. Bonabeau, G. Theraulaz, and J.-L. Deneubourg. "Quantitative study of the fixed threshold model for the regulation of division of labour in insect societies". In: *Proc. R. Soc. Lond. B* 263.1376 (1996), pp. 1565–1569.

[BTD98] E. Bonabeau, G. Theraulaz, and J.-L. Deneubourg. "Fixed response thresholds and the regulation of division of labor in insect societies". In: *Bulletin of Mathematical Biology* 60.4 (1998), pp. 753–807.

[CDLN14] A. Cornejo, A. R. Dornhaus, N. A. Lynch, and R. Nagpal. "Task Allocation in Ant Colonies". In: *Distributed Computing - 28th International Symposium, DISC 2014, Austin, TX, USA, October 12-15, 2014. Proceedings.* 2014, pp. 46–60. URL: https://doi.org/10.1007/978-3-662-45174-8_4.

[DLM+18] A. R. Dornhaus, N. A. Lynch, F. Mallmann-Trenn, D. Pajak, and T. Radeva. "Self-Stabilizing Task Allocation In Spite of Noise". In: *CoRR* abs/1805.03691 (2018). arXiv: 1805.03691. URL: http://arxiv.org/abs/1805.03691.

[DPKW12] A. Duarte, I. Pen, L. Keller, and F. J. Weissing. "Evolution of self-organized division of labor in a response threshold model". In: *Behavioral Ecology and Sociobiology* 66.6 (2012), pp. 947–957.

[LTKF12] P. Lichocki, D. Tarapore, L. Keller, and D. Floreano. "Neural networks as mechanisms to regulate division of labor". In: *The American Naturalist* 179.3 (2012), pp. 391–400.

[MWCG16] B. Meyer, A. Weidenmüller, R. Chen, and J. Garcia. "Collective Homeostasis and Time-resolved Models of Self-organised Task Allocation". In: *Proceedings of the 9th EAI International Conference on Bio-inspired Information and Communications Technologies (formerly BIONETICS).* ICST (Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering). 2016, pp. 469–478.

[MU05] M. Mitzenmacher and E. Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Cambridge University Press, 2005.

[PGG96] S. W. Pacala, D. M. Gordon, and H. Godfray. “Effects of social group size on information transfer and task allocation”. In: *Evolutionary Ecology* 10.2 (1996), pp. 127–165.

[PM98] R. E. Page Jr and S. D. Mitchell. “Self-organization and the evolution of division of labor”. In: *Apidologie* 29.1-2 (1998), pp. 171–190.

[Wil76] E. O. Wilson. “Behavioral discretization and the number of castes in an ant species”. In: *Behavioral Ecology and Sociobiology* 1.2 (1976), pp. 141–154.