SPIKING NEURAL NETWORKS: AN ALGORITHMIC PERSPECTIVE

Nancy Lynch, Cameron Musco and Merav Parter

Massachusetts Institute of Technology, EECS.
Weizmann Institute of Science
BDA 2017.
Based on work in:

- *Computational Tradeoffs in Biological Neural Networks: Self-Stabilizing Winner-Take-All Networks*. ITCS 2017.
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Full versions are available at: [cameronmusco.com](http://cameronmusco.com)
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We focus on fixed networks and do not (yet) consider how they are learned. Our tasks are basic computational primitives rather than more complex pattern recognition goals.
GUIDING QUESTIONS

• How do biological features affect computability, runtime tradeoffs, and algorithm design?

• Is there interesting theory beyond what is known for other well-studied models of computation. E.g. deterministic threshold circuits (perceptrons), Boltzmann machines, distributed networks with message passing, etc.?

• Can this theory say anything interesting about computation in real neural networks? E.g. role of noise and randomness, roll of inhibition and excitation, recurring design patterns.

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All neurons are strictly inhibitory or excitatory – i.e. \( w(u, v) \geq 0 \) for all \( v \) or \( w(u, v) \leq 0 \) for all \( v \).
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Ignore many other biological features. E.g. refractory period, spike propagation delay, memory, noise on synapses etc. Some can be simulated in our model.
• $n$ input neurons $X$ each either always firing or not firing.
• $m$ output neurons $Y$. 

COMPUTATIONAL PROBLEMS IN OUR MODEL
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Questions so far?
Example Problem 1
**Binary WTA problem:** Want to converge to a single firing output, which corresponds to a firing input.
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- Neural leader election. Very heavily studied in computational neuroscience.
- Used in perceptual attention, competitive learning, etc. Powerful ‘nonlinear’ primitive [Maass ’99]
SIMPLE SOLUTION WITH TWO INHIBITORS

Main idea:
Inhibitors facilitate competition (or lateral inhibition) between inputs, leading to a single 'winner'.

- **Convergence inhibitor** $z_c$ fires whenever there are $\geq 2$ competing outputs and causes any competing output to stop firing at time $t + 1$ with probability $1/2$. 

![Diagram showing the simple solution with two inhibitors](image-url)
Main idea: Inhibitors facilitate competition (or lateral inhibition) between inputs, leading to a single ‘winner’.
• **Stability** inhibitor $z_s$ fires whenever there are $\geq 1$ competing outputs and prevents any output that didn’t fire at time $t$ from firing at time $t + 1$. 
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• **Convergence** inhibitor $z_c$ fires whenever there are $\geq 2$ competing outputs and causes any competing output to stop firing at time $t + 1$ with probability $1/2$. 
• Roughly $1/2$ of competing outputs stop firing at each time step. With constant probability there is some time $t \leq \log n$ such that exactly one output fires at time $t$.

• After time $t$, this distinguished output continues to fire. Just $z_s$ fires, preventing all other outputs from firing.
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- Can be used to solve non-binary WTA. Goal here is to select the input with the highest firing rate.
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1. Inhibitors fall into two classes – convergence and stability neurons. Inhibition is often viewed as a stability mechanism in the brain. In our networks, it has two roles: maintaining stability and driving computation.

2. Inhibitors behave nearly deterministically. Randomness is used solely for symmetry breaking between outputs. Highlights dual nature of randomness – can be a powerful computational resource but can also slow down computation by leading to noisy behavior.
Example Problem 2
**Similarity Testing:** Given two input firing patterns $X_1$ and $X_2$, distinguish whether $X_1 = X_2$ or if they are far from being equal. I.e. if $d(X_1, X_2) \geq \epsilon n$. 
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- Natural sub-problem for pattern recognition and other tasks.
Simple (non-neural) sublinear time algorithm:

Sample $O(\log n \epsilon)$ random positions and check if $X_1$ and $X_2$ match at these positions. If $X_1 = X_2$, then $S_1 = S_2$. If $d(X_1, X_2) \geq \epsilon n$, then $S_1 \neq S_2$ with high probability.
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\[ X_1 = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \end{bmatrix} \]

\[ X_2 = \begin{bmatrix} x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \end{bmatrix} \]

Random sample

\[ S_1 = \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & \cdots & s_{1,\log n} \end{bmatrix} \]

\[ S_2 = \begin{bmatrix} s_{2,1} & s_{2,2} & s_{2,3} & \cdots & s_{2,\log n} \end{bmatrix} \]
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If $X_1 = X_2$, then $S_1 = S_2$. If $d(X_1, X_2) \geq \epsilon n$, the $S_1 \neq S_2$ with high probability.
Equality check of $S_1$ and $S_2$ is straightforward.

Sampling random positions requires an indexing module: given an index encoded by the firing pattern of a set of neurons, select the appropriate value of $X_1$ or $X_2$.

After convergence, the output neuron should fire continuously if and only if $X(Z)$ is firing.

Simulates an excitatory connection from $X(Z)$ to $y$. 
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![Diagram of a neural network with nodes labeled $z_1, z_2, \ldots, z_{\log n}, x_1, x_2, x_3, \ldots, x_n$, and output node $y$. The diagram shows connections between these nodes, indicating how the neurons interact to perform the required computations.]
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- Uses information contained in a small set of neurons (the index) to access information from a much larger data store $X$.
- This seems to be an important primitive in many computations beyond our similarity testing application. E.g. a smell or sight triggering a memory.
Theorem

For any $t \leq \sqrt{n}$, there is an SNN solving the indexing problem with $O(n/t)$ auxiliary neurons that converges in $t$ time steps with high probability. For $t = \sqrt{n}$, the circuit uses $O(\sqrt{n})$ auxiliary neurons.
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- Gives an $O\left(\frac{\sqrt{n} \log n}{\epsilon}\right)$ (i.e., sublinear) sized circuit for the similarity testing problem.
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• Also separates our model from sigmoidal gates with real valued outputs which can implement indexing with $O(\sqrt{n})$ neurons converging in $O(1)$ steps.
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• Is general indexing machinery actually implemented in the brain?
• Our similarity testing algorithm is a simple application of randomized compression.
  • Other randomized compression schemes like Johnson-Lindenstrauuss projection have been considered as possible neural algorithms.
  • To what extent are these schemes implemented via random connectivity and to what extent do they require indexing operations?
Concrete Next Steps:
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- $k$-WTA, better understanding of WTA with non-binary inputs, sparse coding/renaming

- More biologically plausible models: refractory period, history, asynchrony, learning and dynamic synapse weights

- Theoretical abstractions that let us handle biological complexity.

- What features of our model can be generalized? E.g. can we prove results for a wider class of activation functions beyond the sigmoid?
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High-Level Directions

• How do networks for simple computational primitives arise? Can they be ‘learned’? Are they preprogramed in some way?

• How do fixed network motifs such as WTA and similarity testing circuits interact with more flexible ‘learning’ networks?

• More generally, would like to develop a theory for composing spiking neural networks to solve complex problems.
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Thanks!