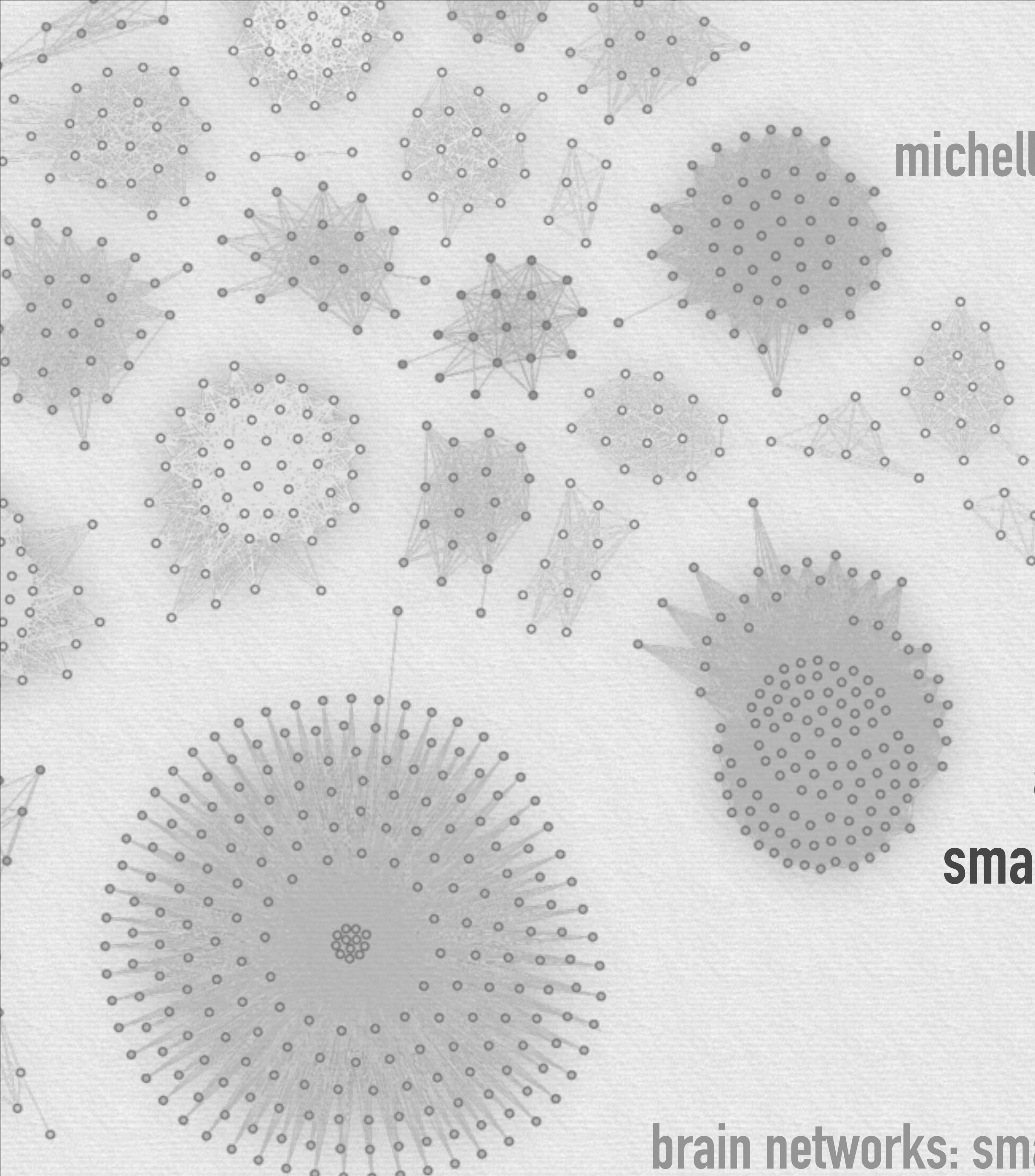
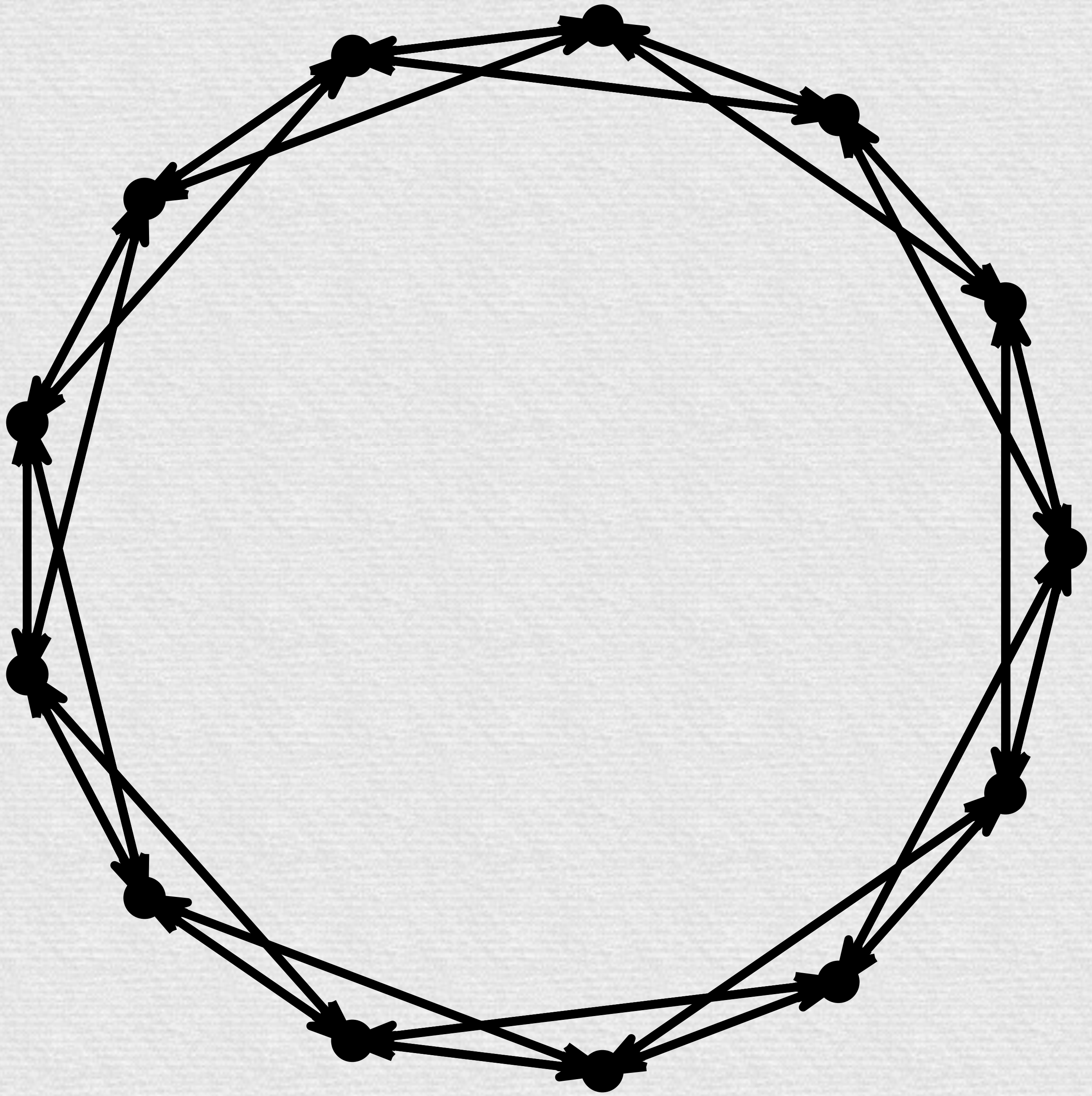


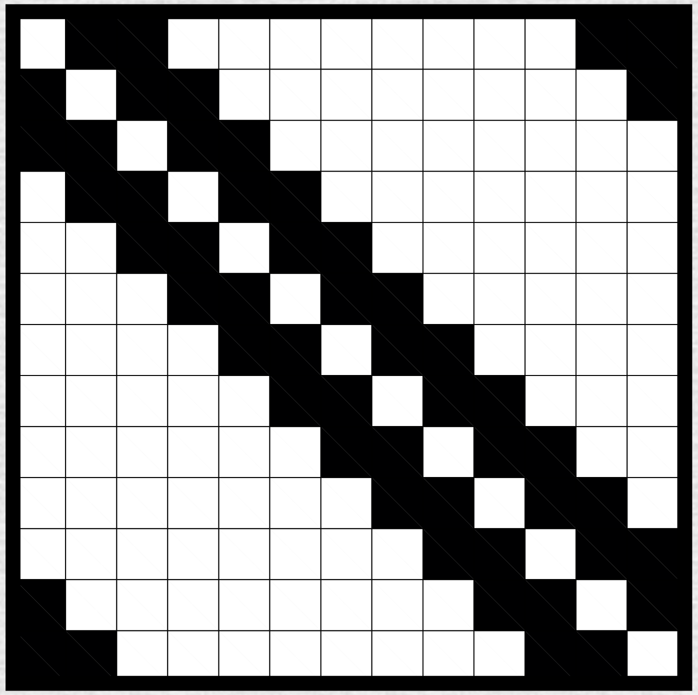
michelle rudolph-lilith
*lyle e muller

an algebraic approach to small-world network models

brain networks: small-worlds, after all?

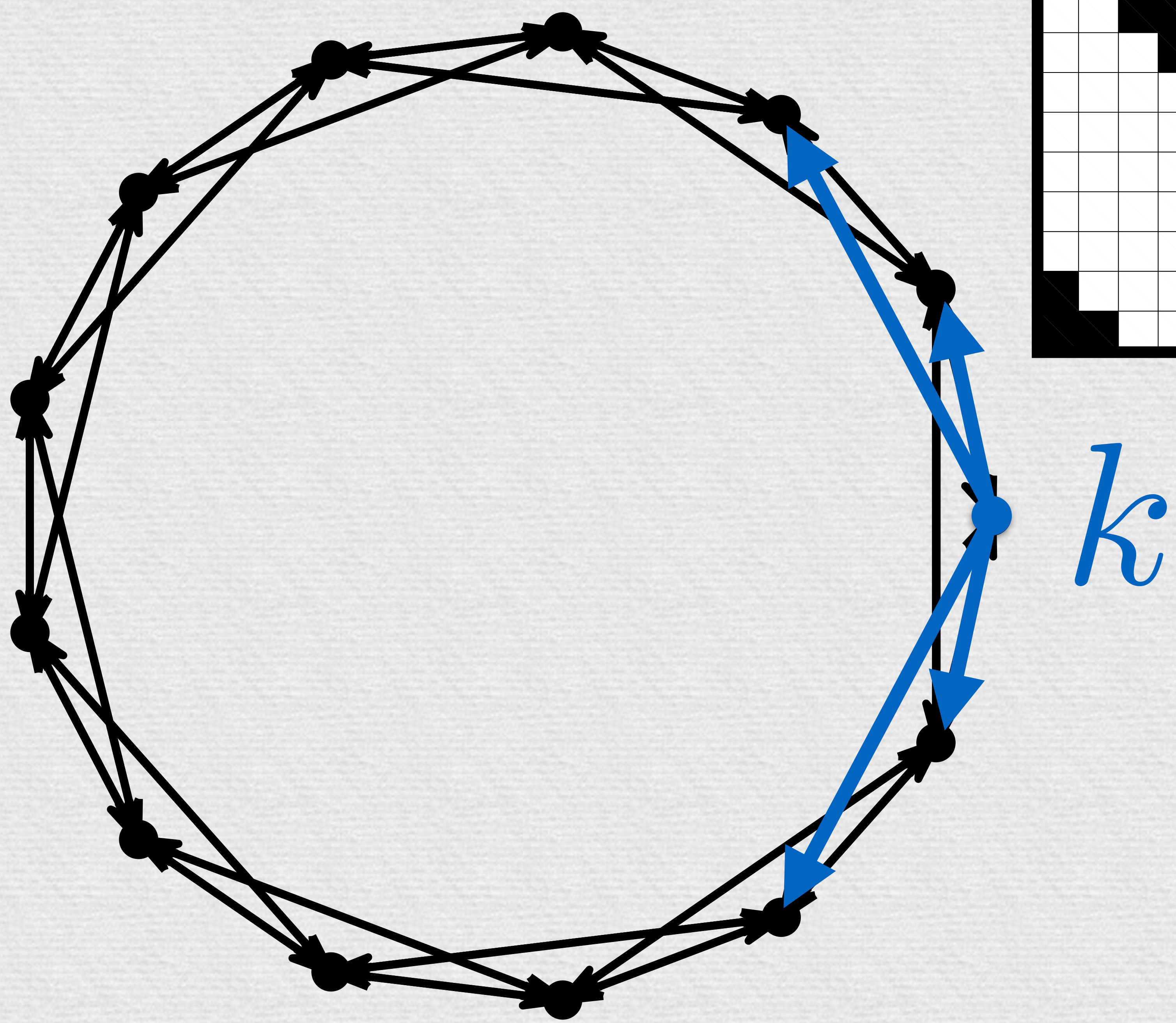
Ring Graph

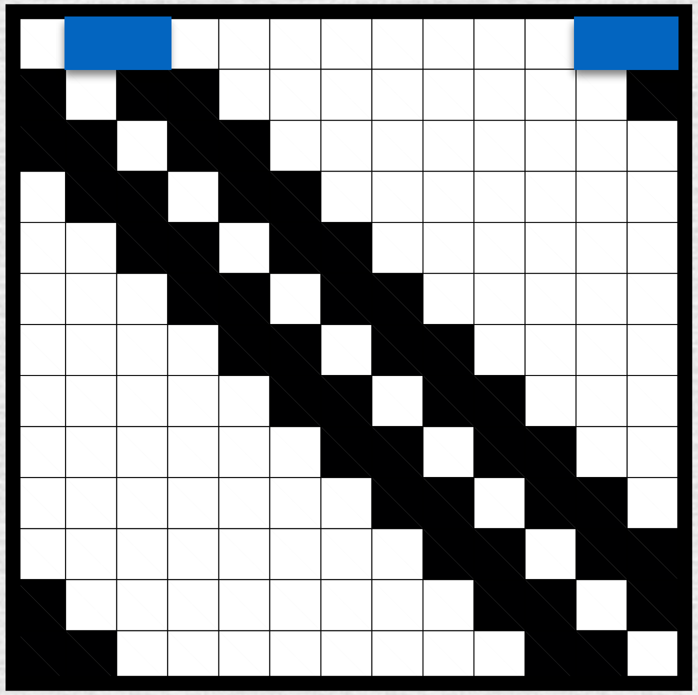




$$\mathfrak{G}_{RG}$$

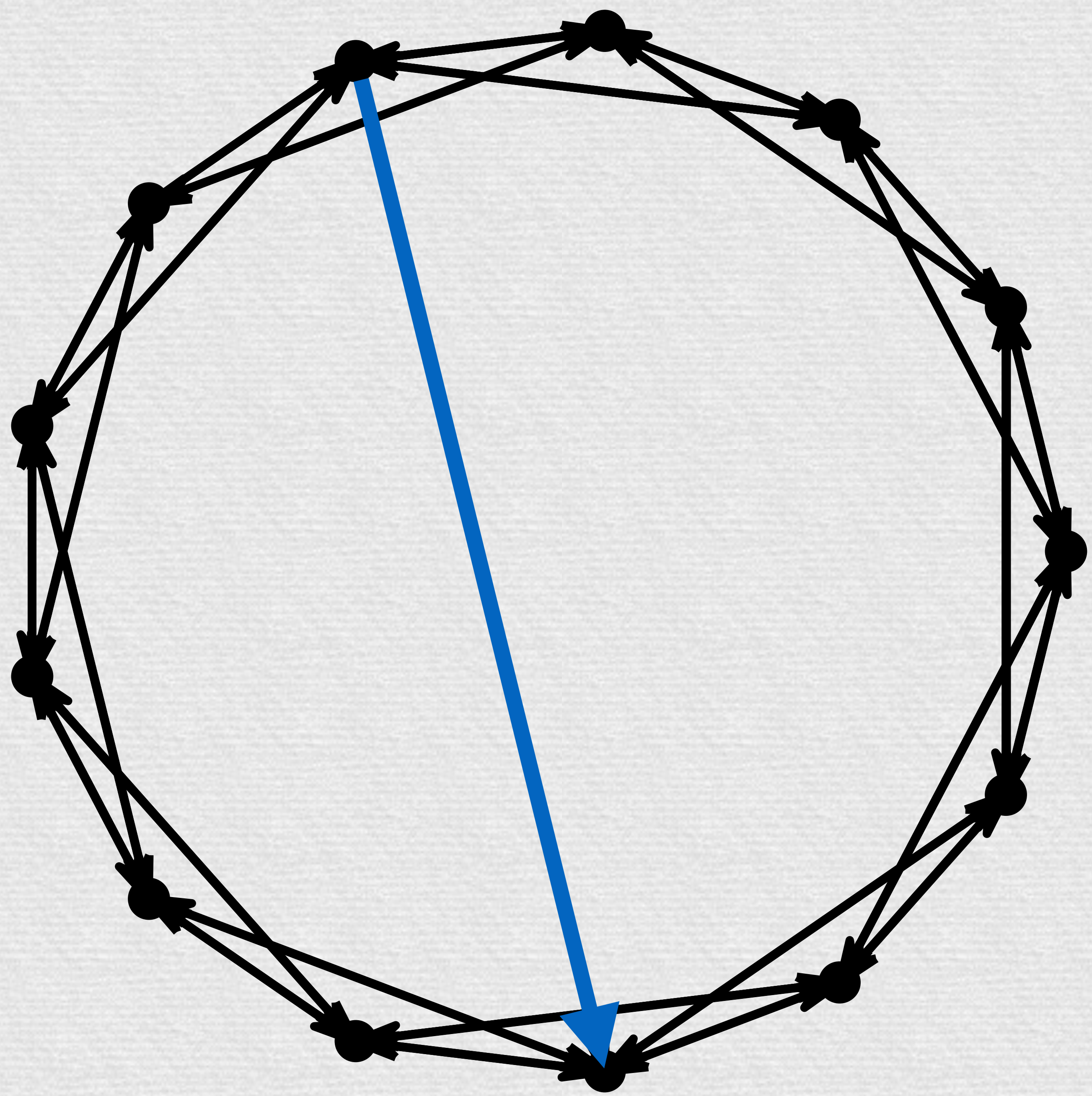
Ring Graph

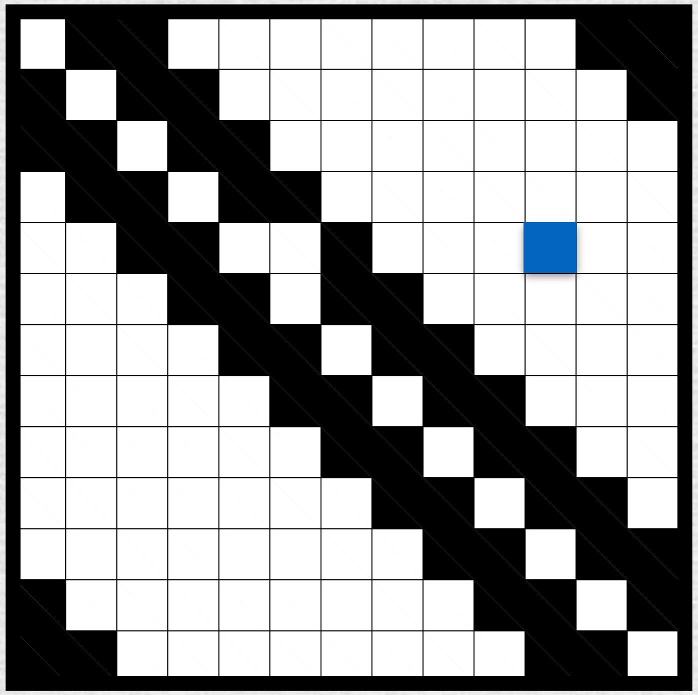




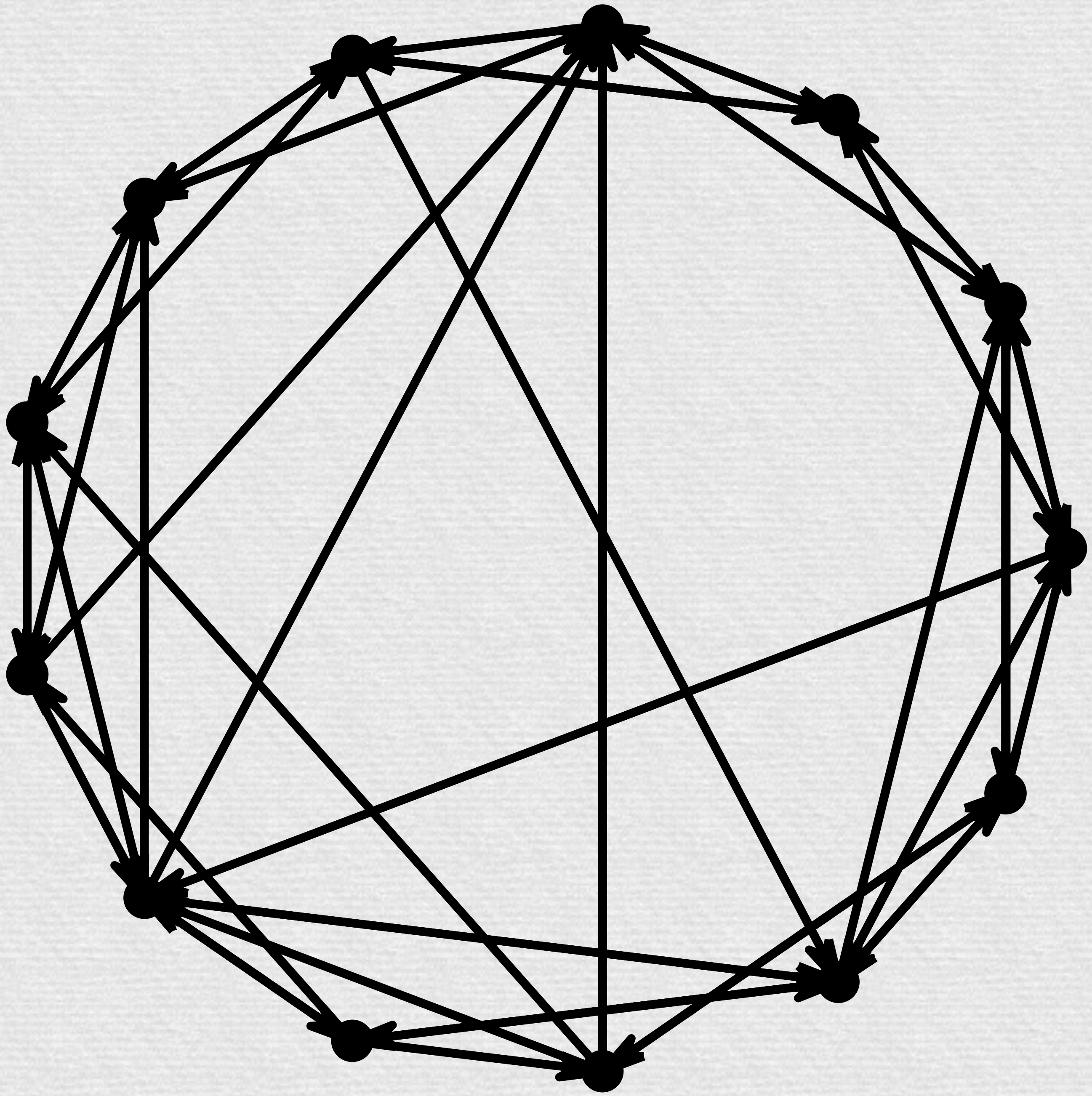
$$N_E^{RG} = 2kN_N$$

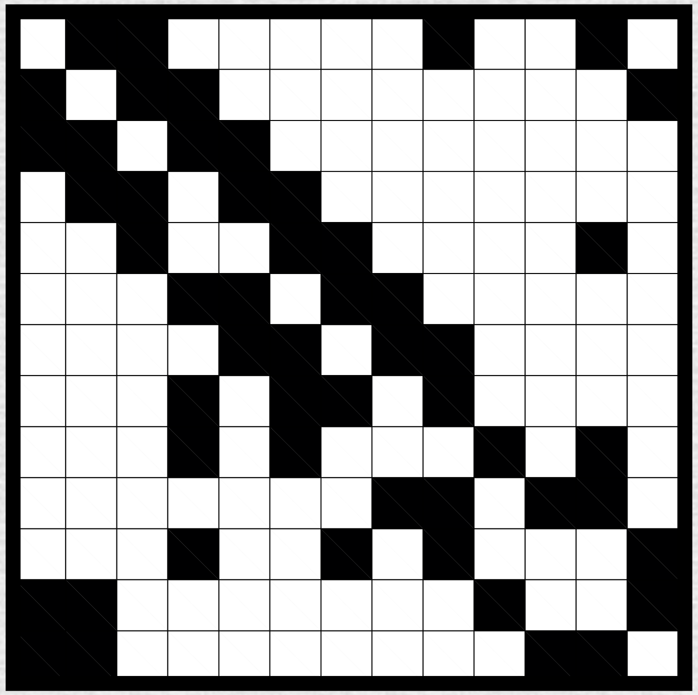
Rewiring



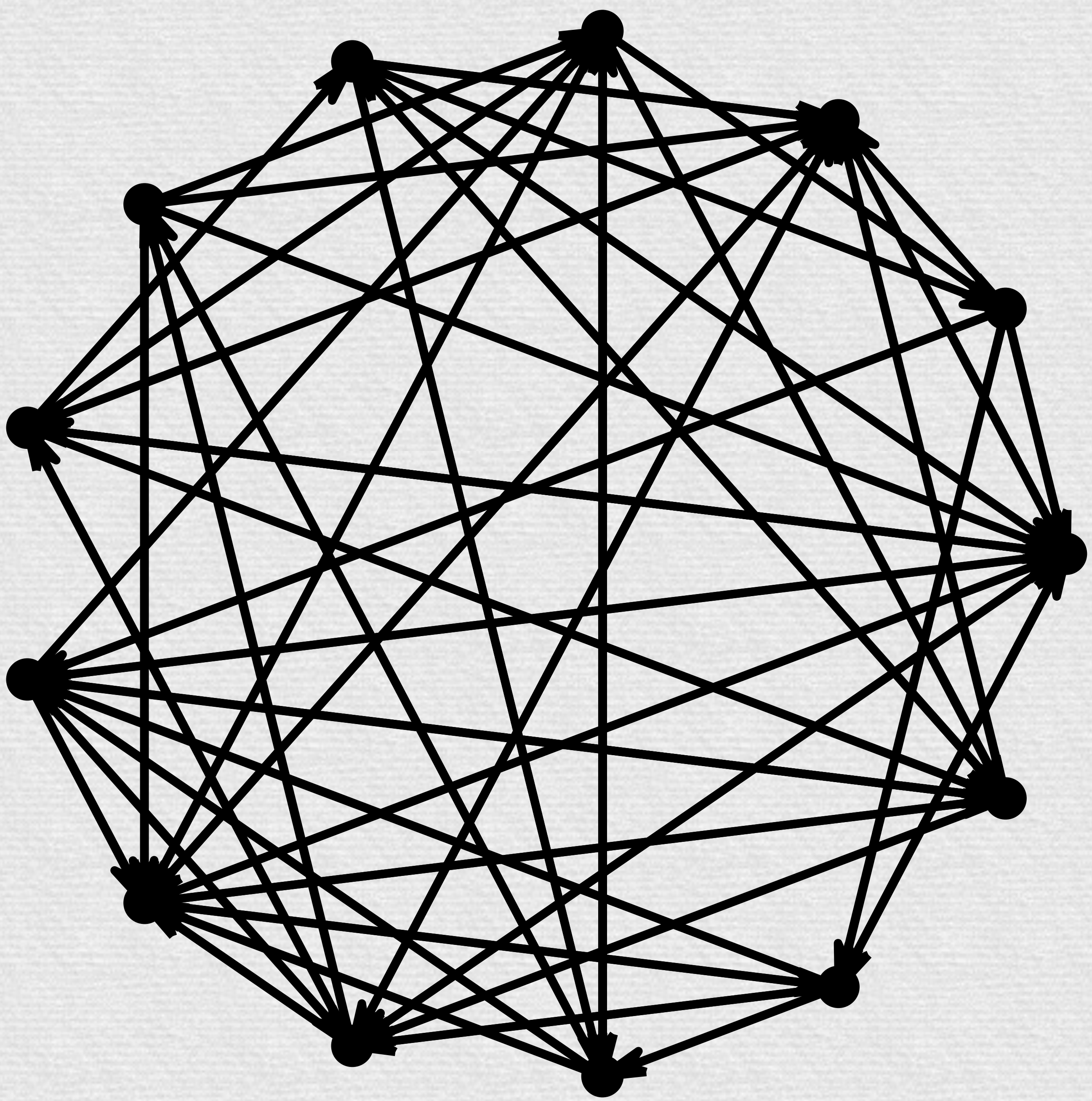


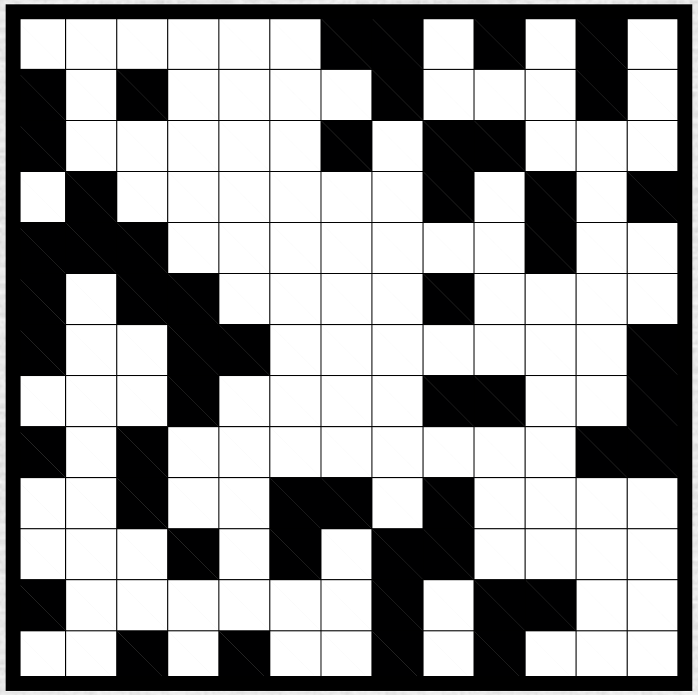
Rewiring



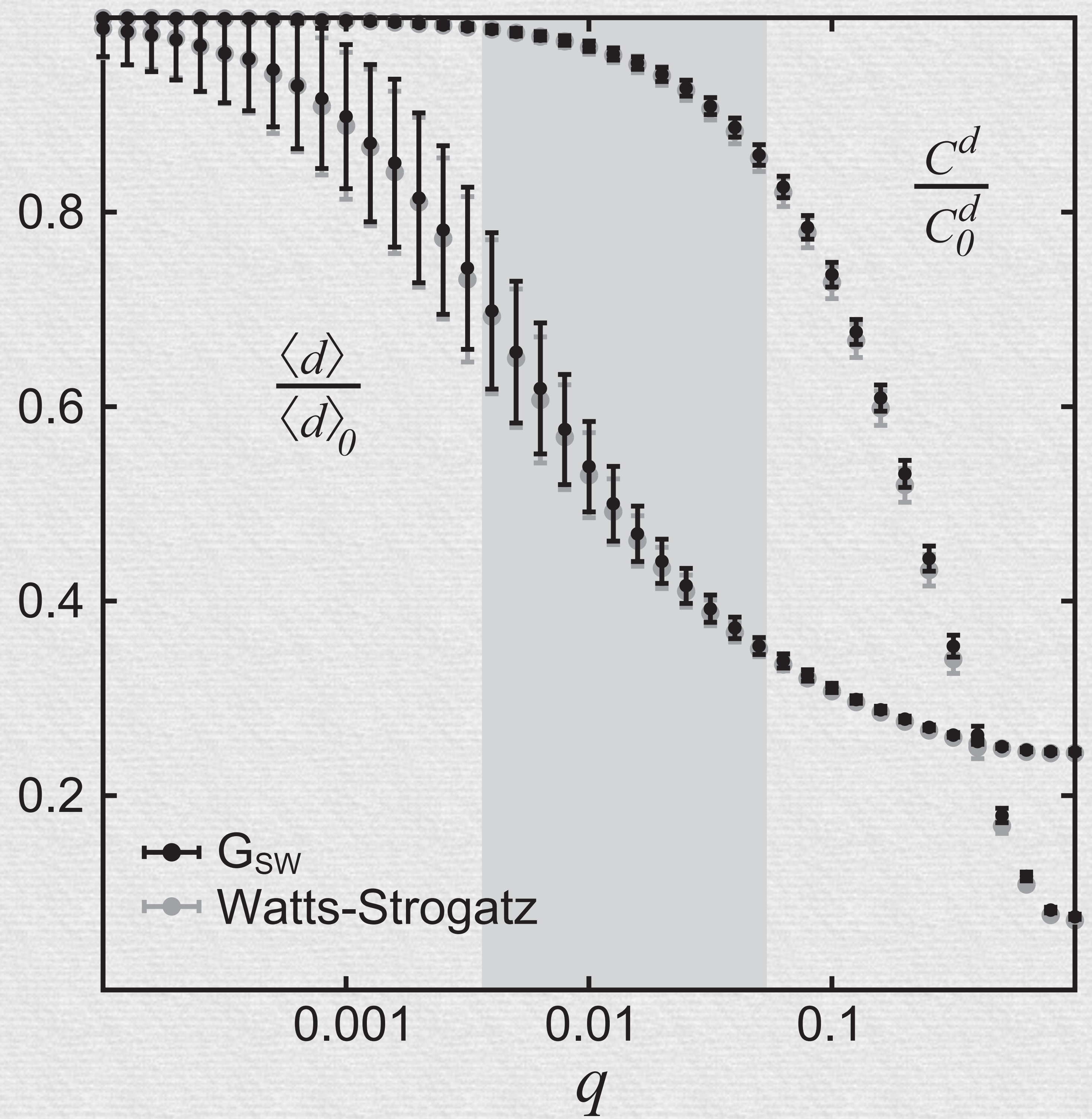


Random Graph





$\mathfrak{G}_{SW}$



$\mathfrak{G}_{SW}$

- Consider random, uniform rewiring on a ring graph.
- Note that this is equivalent to starting with a "reduced" ring graph with

  $$2kN_N - q2kN_N(N_N - 2k - 1)/(N_N - 1) \text{ edges}$$

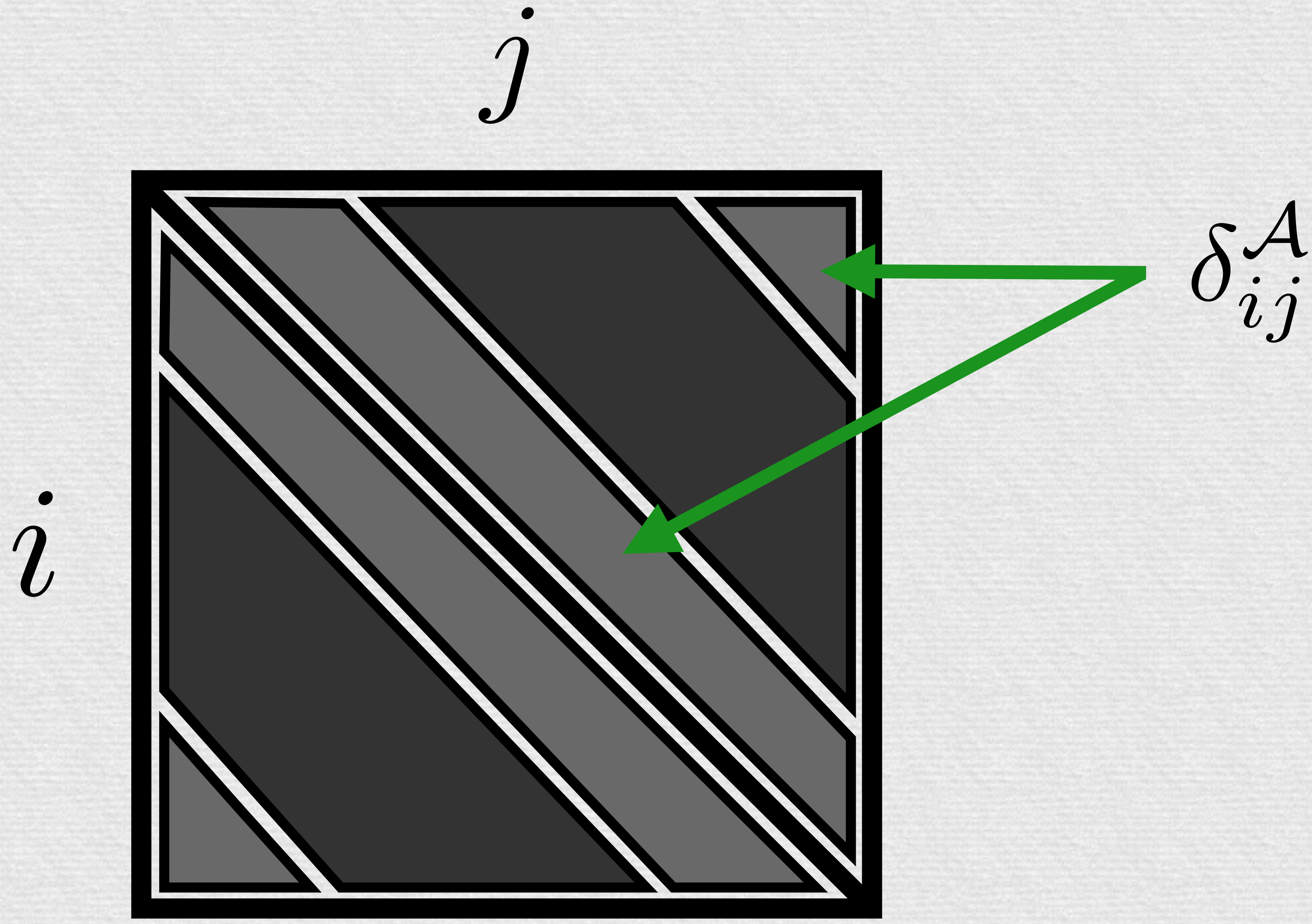
  uniform randomly distributed across the $2kN_N$ edges of $\mathfrak{G}_{RG}$, and distributing the remaining

  $$q2kN_N(N_N - 2k - 1)/(N_N - 1) \text{ edges}$$

  outside $\mathfrak{G}_{RG}$.

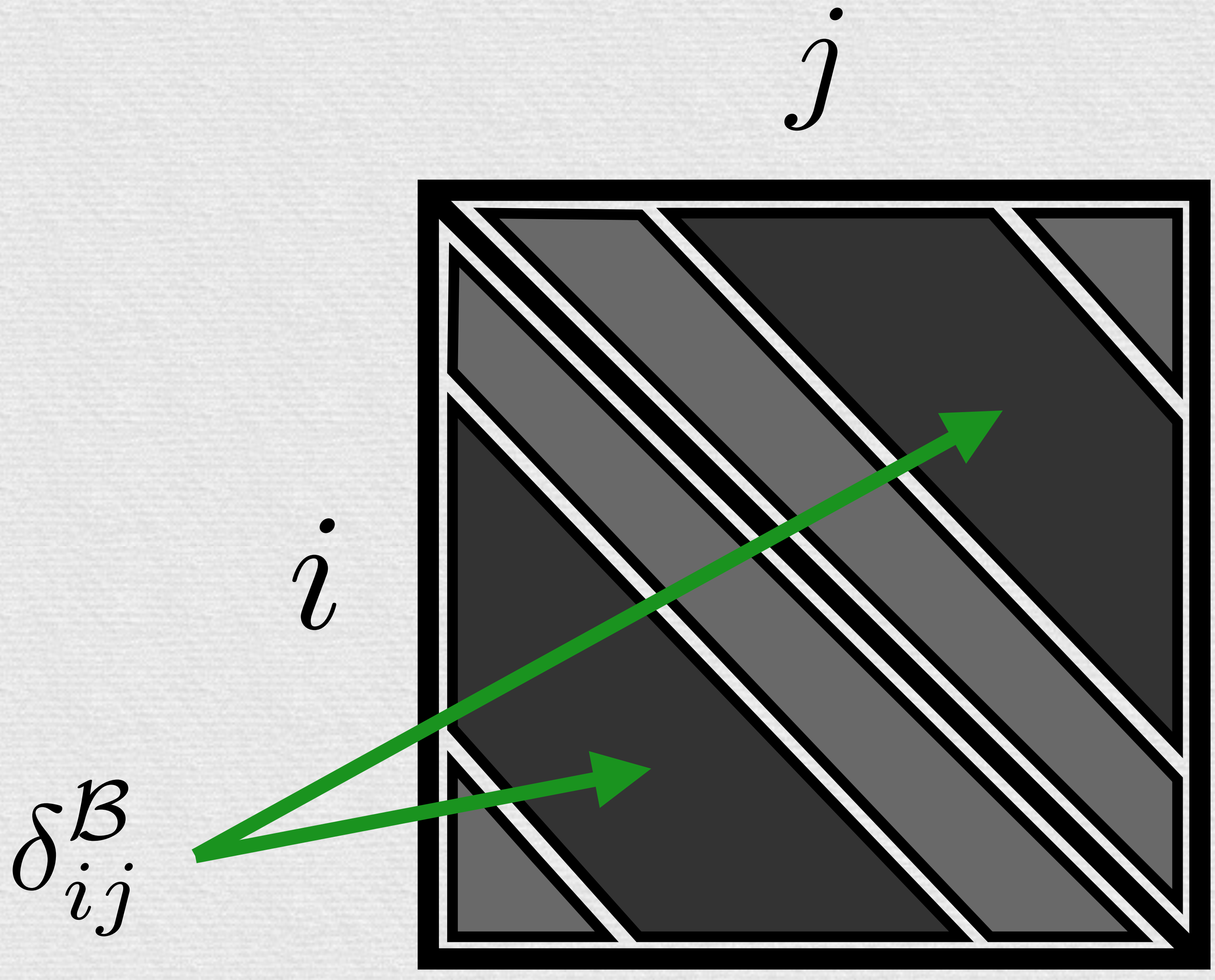
[Rudolph-Lilith & Muller, *Physical Review E.* **89** (2014): 012812]

$\delta_{ij}^{\mathcal{A}}$



$$\mathcal{A} = \Big(\{i,j\}, i,j \in [1, N_N] : \big((i-k \leq j \leq i+k) \wedge (i \neq j)\big) \vee (i + N_N - k \leq j) \vee (j + N_N - k \leq i)\Big)$$

$\delta_{ij}^{\mathcal{B}}$



$$\mathcal{B} = \Big(\{i,j\}, i,j \in [1, N_N] : (i+k < j < i + N_N - k) \vee (j+k < i < j + N_N - k)\Big)$$

Random Annihilation Operator

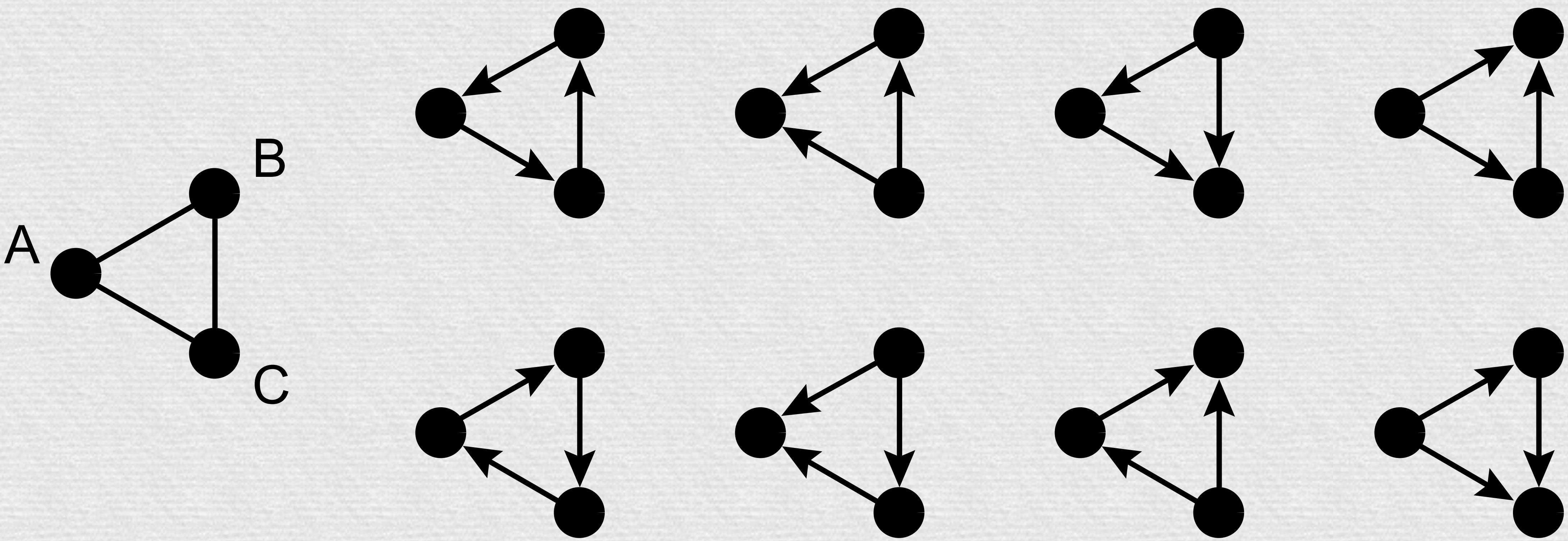
$$\hat{r}^p(x) = \begin{cases} x & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

$$\sum^{n} \hat{r}^p(x) = npx$$

Algebraic Adjacency Matrix

$$a_{ij}^{SW} = (1 - \hat{r}_{ij}^{p_{\mathcal{A}}})\delta_{ij}^{\mathcal{A}} + \hat{r}_{ij}^{p_{\mathcal{B}}}\delta_{ij}^{\mathcal{B}}$$

Clustering Coefficient



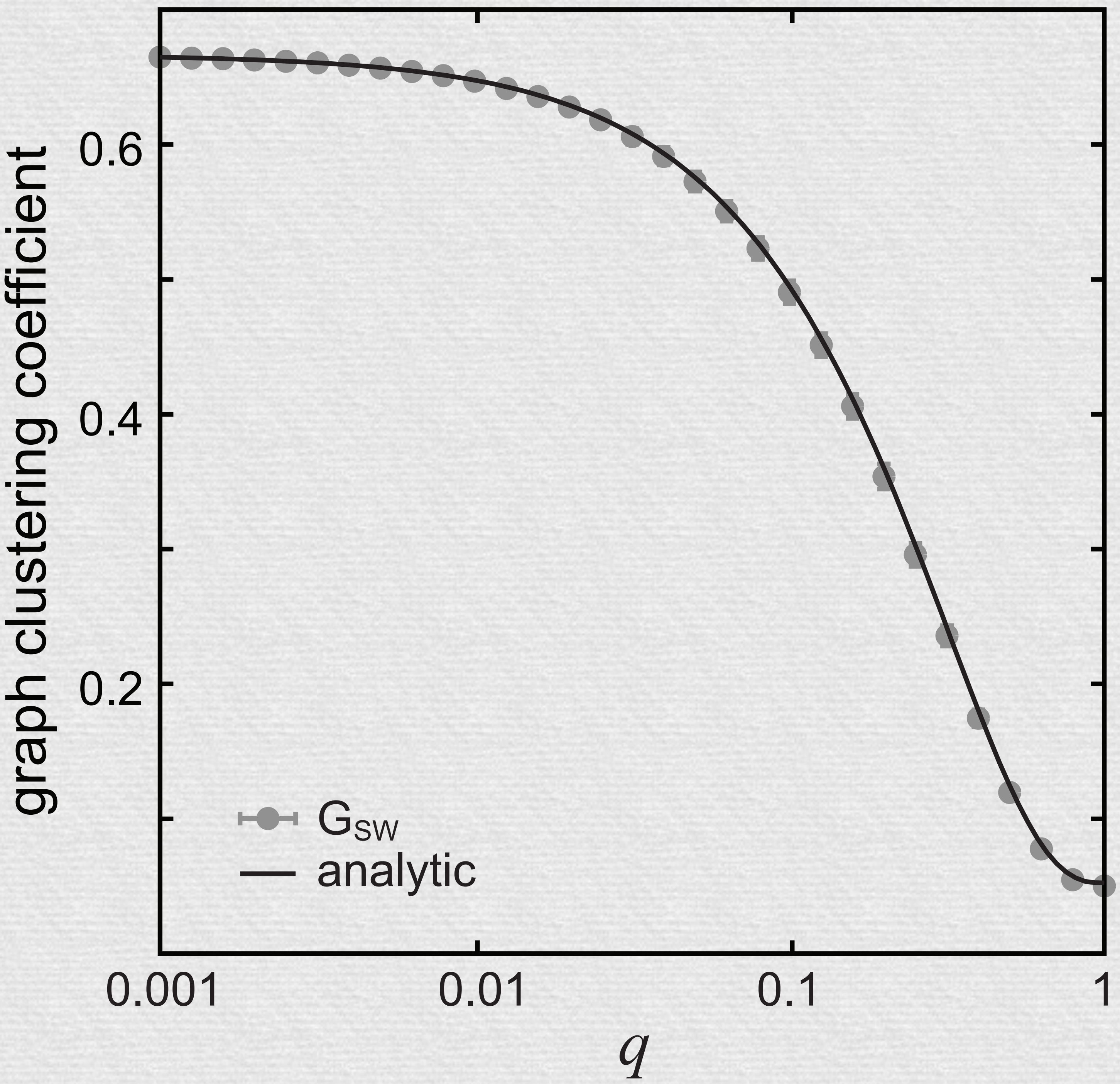
$$C^d = \frac{1}{\mathcal{N}} \sum_{i,j,h=1}^{N_N} (a_{ij} + a_{ji})(a_{ih} + a_{hi})(a_{jh} + a_{hj})$$

Clustering Coefficient

$$
\begin{aligned}
C^{d\,SW} \quad = \quad & \frac{64}{\mathcal{N}^{SW}(N_N-1)^3} \\
& \times \Bigg\{ \Big(N_N - 1 - q(N_N - 2k - 1)\Big)^3 \Big(k^3 + \sum_{m=1}^{N_N-1} a_m^3(N_N,k)\Big) \\
& \quad + 3kq\Big(N_N - 1 - q(N_N - 2k - 1)\Big)^2 \Big(k^2(N_N - 2k - 1) - \sum_{m=1}^{N_N-1} a_m^2(N_N,k) b_m(N_N,k)\Big) \\
& \quad + 3k^2q^2\Big(N_N - 1 - q(N_N - 2k - 1)\Big) \Big(k(N_N - 2k - 1)^2 + \sum_{m=1}^{N_N-1} a_m(N_N,k) b_m^2(N_N,k)\Big) \\
& \quad + q^3k^3\Big((N_N - 2k - 1)^3 - \sum_{m=1}^{N_N-1} b_m^3(N_N,k)\Big) \Bigg\}
\end{aligned}
$$

[Rudolph-Lilith & Muller, *Physical Review E.* **89** (2014): 012812]

Clustering Coefficient

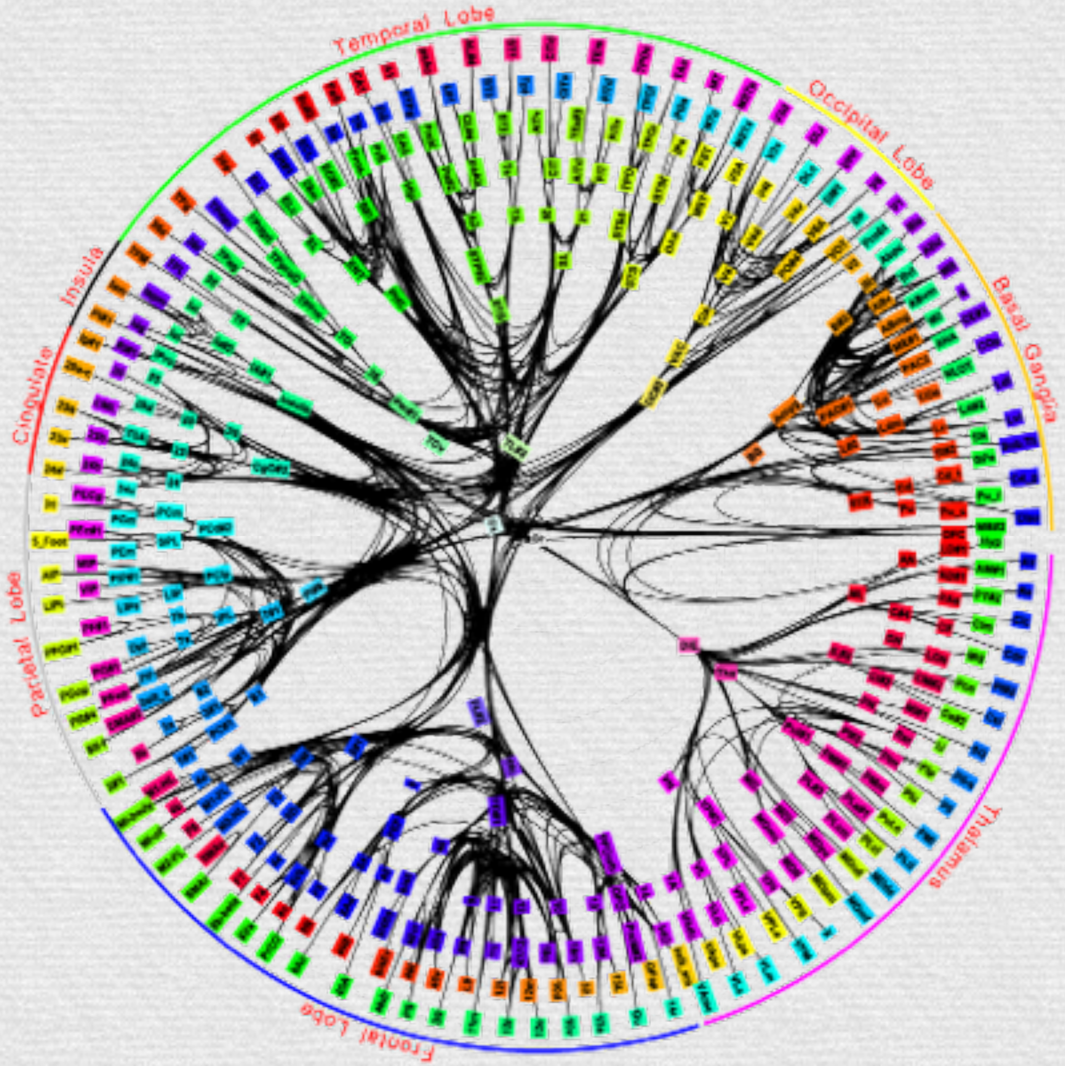


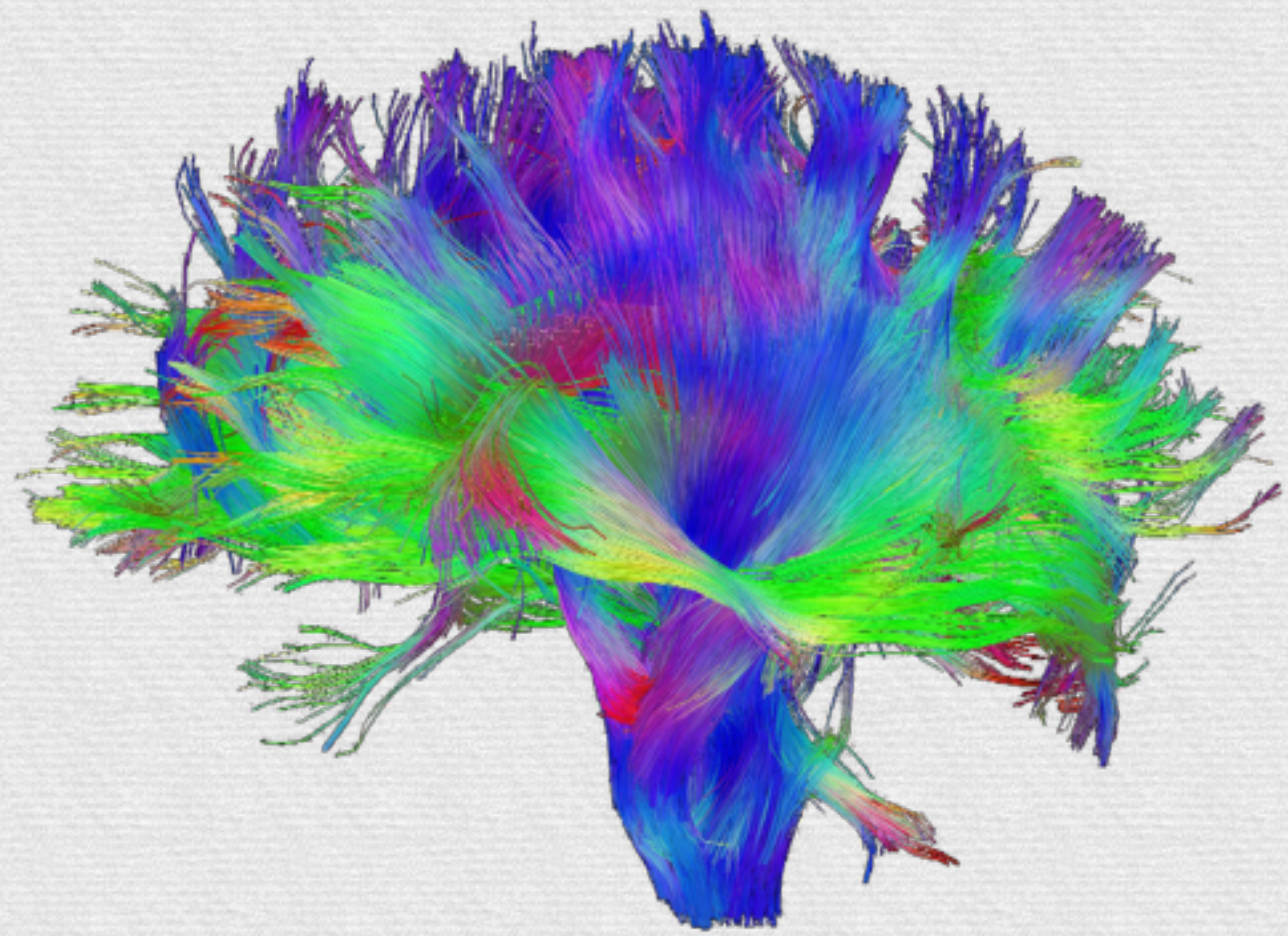
Clustering Coefficient: Scaling Limit

$$C^{d\,SW}\Big|_{N_N \to \infty} = \frac{3(k-1)}{2(2k-1)+q(2-q)}(q-1)^3$$

[cf. Barrat & Weigt, *European Physics Journal B.* **13** (2000): 547-560]

Application to Neural Graphs

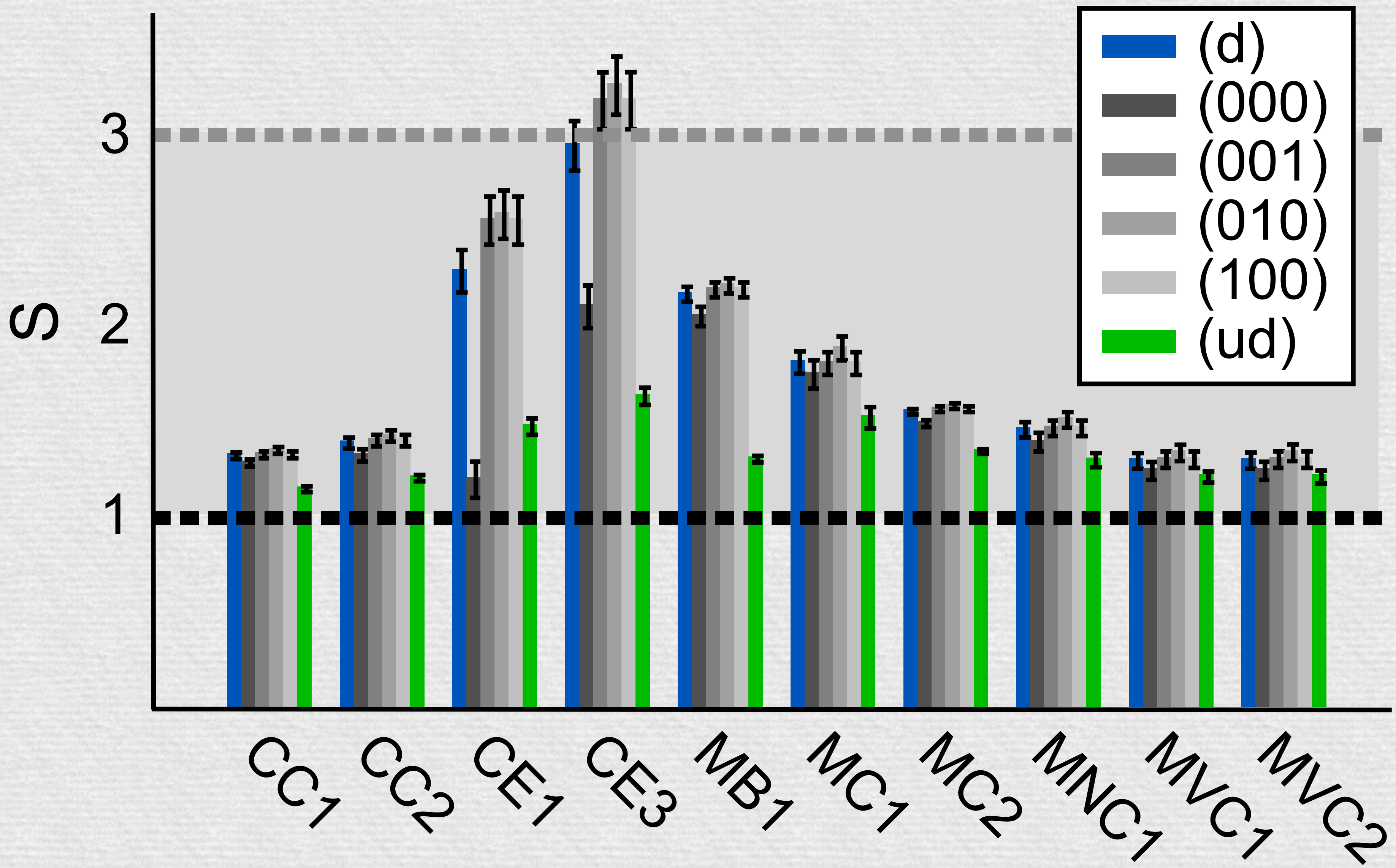




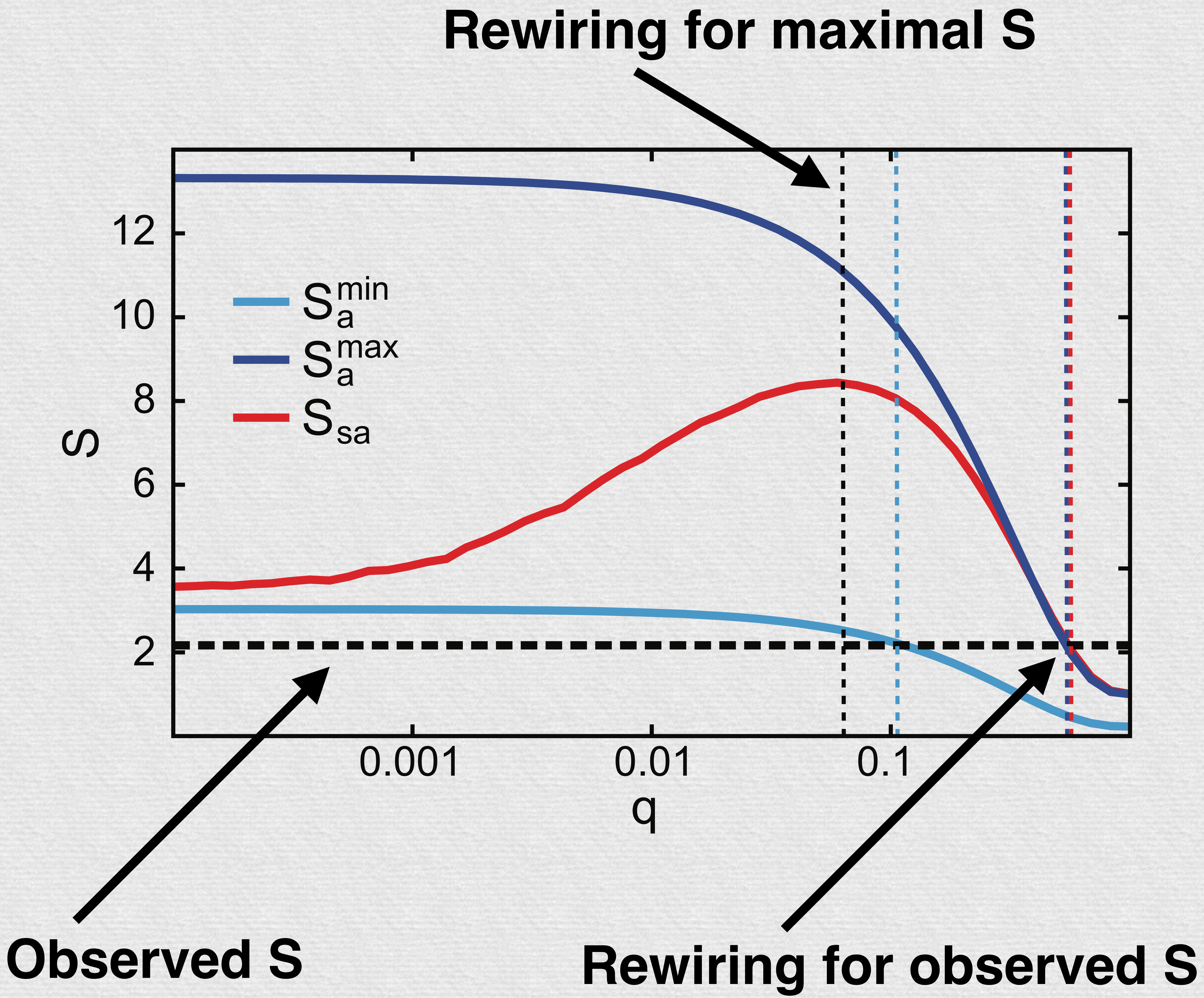
[Muller et al., *New Journal of Physics.* (2014): *basically submitted*]

Small-Worldness Index

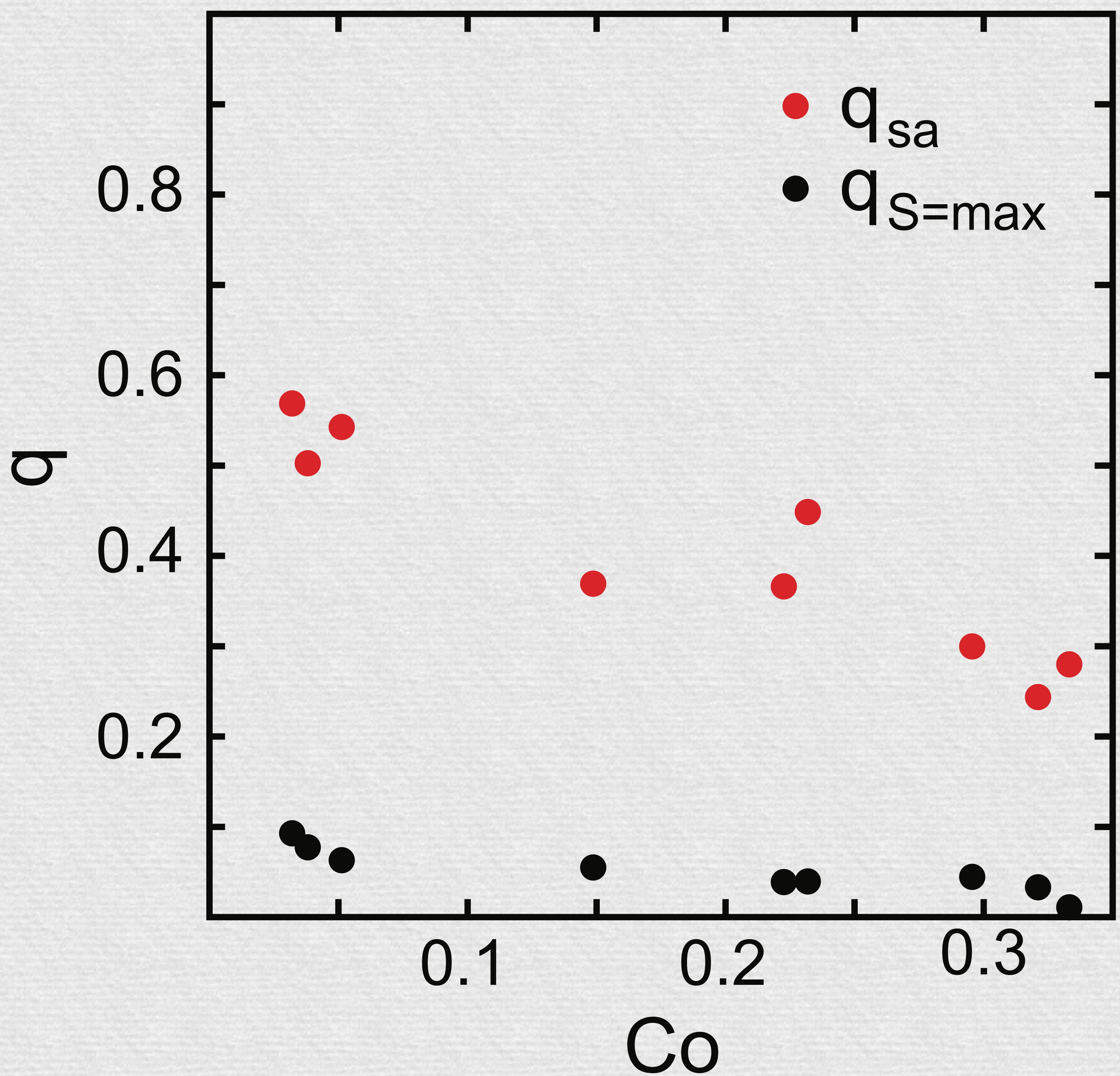
$$S^d = \frac{C^d_{SW}}{C^d_{ER}} \frac{\langle d \rangle_{ER}}{\langle d \rangle_{SW}}$$



Network Small-Worldness



Required Rewiring



Conclusions