A class of Lorenz-like systems

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The transformation of a three-dimensional dynamical system to its differential model can be used to identify different nonlinear dynamical systems that share the same time series of one of its variables. This transformation then can be used to find classes of nonlinear dynamical systems with similar dynamical behavior. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3689438]

The characterization of the algebraic structure of a system of nonlinear ordinary differential equations (ODEs) is an open problem. There exist systems that have a different algebraic structure but have the same dynamical behavior such as the Lorenz and the Wang systems. In the case of some special choices of the parameters of those two systems, they even can produce the exact same time series of one of their variables. On the other hand, there are, e.g., the Lorenz and the Chen systems. These two systems have the same algebraic structure but the dynamics is different. There are more examples like that and a rigorous characterization of all these systems from their algebraic structure would be needed. Here, the transformation of all these systems to jerk form (also called differential form) is used to detect similarities and differences between different systems. This way the equivalence of systems such as the Lorenz and the Wang systems, and the difference between the Lorenz and the Chen systems can be understood. In this paper, there are three published Lorenz-like systems characterized based on their algebraic structure of their jerk form and there are 17 possible Lorenz-like systems shown that have the same jerk form.

I. INTRODUCTION

In 1963, Lorenz[1] pointed out that reductions of the Navier–Stokes partial differential equations to sets of ordinary differential equations assumed a simple polynomial form

$$\frac{dx_i}{dt} = a_{ijk}x_jx_k - b_{ij}x_j + c_i. \tag{1}$$

He further pointed out that if the cubic form $a_{ijk}x_ix_jx_k$ vanishes, then every trajectory becomes trapped inside an ellipse of finite size as $t \to \infty$. If this condition is not met, some initial conditions can wander off to infinity. The Lorenz equations

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= Rx - y - xz, \\ \dot{z} &= -bz + xy \end{aligned} \tag{2}$$

satisfy this condition.

In view of the importance of the class of equations (Eq. (1)) described by Lorenz, it is natural to ask if it is possible to represent experimental time series by equations with this structure. If it is possible, is the representation unique? If not, what are the degrees of freedom in representations of data by equations from this class of equations?

This problem is approached by transforming these equations to a standard form ("differential form" or "jerk form"). For a three-dimensional phase space, this standard form has the structure

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= F(X,Y,Z). \end{aligned} \tag{3}$$

The first variable X is some function of the phase space variables: $X = \phi(x,y,z) = \phi(x_1,x_2,x_3)$. Many different sets of equations of the type shown in Eq. (1) can map to the same differential form, even with the same parameter values. The number of distinct sets of equations with the same image differential equation is a measure of the nonuniqueness inherent in attempts to model data using equations of a particular class.[2]

The paper is organized as follows: In Sec. II, the class of dynamical systems that are considered here is introduced. Then, the differential model of the Lorenz system is introduced, and the relation of the parameters of the differential model to the original Lorenz system is used to identify the 2-parameter family of Eqs. (11) that share the same time series of $x_1(t)$. This analysis is then used to explain the difference of the Lorenz and the Chen[3] systems. In Sec. III, the general Lorenz-like system is introduced, and the relations between the full general Lorenz-like system and its sub-systems are shown. As example, it is shown that the Wang *et al.*[4] and the Lorenz systems share the same time series of the x_1-variable. The Lorenz, the Chen, and the Wang systems have the same differential model. In Ref. 5, a classification of some Lorenz-like systems was done using feedback circuits. In Sec. III C, it is shown that the Ansatz library analysis yields a corresponding classification of those systems from a purely algebraic point of view. Section IV is the summary and discussion.

II. GENERAL DESCRIPTION OF THE PROBLEM

A. Background

The class of models considered here is a 3D system of ODEs with the right hand sides containing polynomials with up to second order non-linearities (cf. Eq. (1)), which can be written in a general form as

$$\begin{aligned}\dot{x}_i &= a_{i,0} + a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3 + a_{i,4}x_1^2 \\ &\quad + a_{i,5}x_1x_2 + a_{i,6}x_1x_3 + a_{i,7}x_2^2 + a_{i,8}x_2x_3 \\ &\quad + a_{i,9}x_3^2; \quad i = 1,2,3. \end{aligned} \tag{4}$$

Usually, only a small subset of coefficients $a_{i,*}$ is assumed to be nonzero. This subset defines the class of models under consideration. This class has N_m nonzero parameters $a_{i,*}$.

The two questions asked here are (1) can a member of this class of dynamical systems generate a particular time series? and (2) is this member unique or not?

To answer these questions, we investigate the Lorenz class of dynamical systems

$$\begin{aligned}\dot{x}_1 &= a_{1,1}x_1 + a_{1,2}x_2, \\ \dot{x}_2 &= a_{2,1}x_1 + a_{2,2}x_2 + a_{2,6}x_1x_3, \\ \dot{x}_3 &= a_{3,3}x_3 + a_{3,5}x_1x_2. \end{aligned} \tag{5}$$

For this class, $N_m = 7$.

B. Differential model of the Lorenz system

If we choose $X = \phi(x,y,z) = x_1$, the function $F(X, Y, Z)$ in Eq. (3) is

$$\begin{aligned}F(X,Y,Z) &= \alpha_1 X + \alpha_2 X^3 + \alpha_3 Y + \alpha_4 X^2 Y \\ &\quad + \alpha_5 \frac{Y^2}{X} + \alpha_6 Z + \alpha_7 \frac{YZ}{X}, \end{aligned} \tag{6}$$

where the parameters α_i of the differential embedding are related to the parameters $a_{i,*}$ of the Lorenz model by

$$\begin{aligned}\alpha_1 &= (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})a_{3,3}, \\ \alpha_2 &= -a_{1,1}a_{2,6}a_{3,5}, \\ \alpha_3 &= -(a_{1,1} + a_{2,2})a_{3,3}, \\ \alpha_4 &= a_{2,6}a_{3,5}, \\ \alpha_5 &= -a_{1,1} - a_{2,2}, \\ \alpha_6 &= a_{1,1} + a_{2,2} + a_{3,3}, \\ \alpha_7 &= 1. \end{aligned} \tag{7}$$

The differential model has $N_d = 7$ nonzero parameters α_i. The function $F(X, Y, Z)$ in Eq. (3) has two singularities for α_5 and α_7 that were introduced by the transformation to jerk form.

C. Existence of a solution

We now address the first question: Is it possible to fit a time series using a set of equations of the form Eq. (5)? We first attempt to fit the time series to the canonical differential form Eq. (6) by fitting the $N_d = 7$ parameters α_i. If a suitable fit can be found, then it is possible to represent the solution in terms of the Lorenz form Eq. (5) provided the relations (cf. Eq. (7)) can be inverted.

D. Uniqueness of a solution

If a solution can be found, it may not be unique. It is typical that if some set of parameters $a_{i,*}$ satisfies the inverse transformation (7), then a new set of parameters $\tilde{a}_{i,*}$ also satisfies the inverse transformation. The new parameters are related to the original set by a scaling transformation

$$a_{i,*} \rightarrow \tilde{a}_{i,*} = \lambda^{p(i,*)} a_{i,*} \tag{8}$$

and leave the values of the coefficients α_i unchanged.

The simplest way to determine these scaling relations, specifically the set of exponents $p(i,*)$, is to note that each coefficient α_r is a sum of products of powers of the original model parameters $a_{i,*}$. Take the logarithms of these nonlinear product functions, construct the appropriate coefficient matrix and look for the null space.

The scaled version of the inverse transformation (7) with $a_{i,*} \rightarrow \lambda_{i,*} a_{i,*}$ is

$$\begin{aligned}\alpha_1 &= a_{1,1}a_{2,2}a_{3,3}\lambda_{1,1}\lambda_{2,2}\lambda_{3,3} \\ &\quad - a_{1,2}a_{2,1}a_{3,3}\lambda_{1,2}\lambda_{2,1}\lambda_{3,3}, \\ \alpha_2 &= -a_{1,1}a_{2,6}a_{3,5}\lambda_{1,1}\lambda_{2,6}\lambda_{3,5}, \\ \alpha_3 &= -a_{1,1}a_{3,3}\lambda_{1,1}\lambda_{3,3} - a_{2,2}a_{3,3}\lambda_{2,2}\lambda_{3,3}, \\ \alpha_4 &= a_{2,6}a_{3,5}\lambda_{2,6}\lambda_{3,5}, \\ \alpha_5 &= -a_{1,1}\lambda_{1,1} - a_{2,2}\lambda_{2,2}, \\ \alpha_6 &= a_{1,1}\lambda_{1,1} + a_{2,2}\lambda_{2,2} + a_{3,3}\lambda_{3,3}, \\ \alpha_7 &= 1. \end{aligned} \tag{9}$$

To leave α_i in Eq. (7) unchanged, the scale factors have to be one (e.g., for α_1: $\lambda_{1,1}\lambda_{2,2}\lambda_{3,3} = 1$ and $\lambda_{1,2}\lambda_{2,1}\lambda_{3,3} = 1$). Taking the logarithm leads to linear relations (e.g., $\log(\lambda_{1,1}) + \log(\lambda_{2,2}) + \log(\lambda_{3,3}) = 0$ and $\log(\lambda_{1,2}) + \log(\lambda_{2,1}) + \log(\lambda_{3,3}) = 0$). The set of linear relations derived from the 12 terms of Eq. (9) is summarized in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \log(\lambda_{1,1}) \\ \log(\lambda_{1,2}) \\ \log(\lambda_{2,1}) \\ \log(\lambda_{2,2}) \\ \log(\lambda_{2,6}) \\ \log(\lambda_{3,3}) \\ \log(\lambda_{3,5}) \end{pmatrix} = 0. \tag{10}$$

This 12×7 matrix has a two-dimensional null space spanned by the null vectors $(0, 0, 0, 0, -1, 0, 1)$ and $(0, -1, 1, 0, 0, 0, 0)$.

This means that for the Lorenz system, there exists a 2-parameter family of equations

$$\begin{aligned}\dot{x}_1 &= a_{1,1}x_1 + \tfrac{1}{n}a_{1,2}x_2, \\ \dot{x}_2 &= n a_{2,1}x_1 + a_{2,2}x_2 + \tfrac{1}{m}a_{2,6}x_1x_3, \\ \dot{x}_3 &= a_{3,3}x_3 + m a_{3,5}x_1x_2 \end{aligned} \tag{11}$$

that share the same time series $x_1(t)$. One of these two scaling parameters is fixed by the condition that the trilinear form (Lorenz's $a_{ijk}x_ix_jx_k$) vanishes. This places the condition $\frac{1}{m}a_{2,6} + m a_{3,5} = 0$ on the scale factor m, leaving only one independent scale factor n.

Equation (11) can explain the differences between the Lorenz system and the Chen system.[3] In the Lorenz system, the parameter $a_{2,1} = R$ is the most studied bifurcation parameter. For the Chen system, the additional parameter $a_{2,2}$ is changed. Since parameter changes of $a_{2,1}$ and $a_{2,2}$ are not connected by any scaling factor in Eq. (11), the two systems will not share the same time-series $x_1(t)$.

III. GENERAL LORENZ-LIKE SYSTEM

A. Differential model of a general Lorenz-like system

As shown in Refs. 2 and 6 for the example of the Rössler equations, a wider class of systems (wider than Eq. (11)) can exist that share the same differential model and therefore also the same time series. To determine such a class of dynamical systems, the Ansatz library (see Refs. 2 and 6–8 for details) is used to find all 3D systems with constant, linear, and bilinear terms that have the same functional form as the differential model (6). This class of systems is

$$\begin{aligned}\dot{x}_1 &= b_{1,1}x_1 + b_{1,2}x_2,\\ \dot{x}_2 &= b_{2,1}x_1 + b_{2,2}x_2 + b_{2,6}x_1x_3,\\ \dot{x}_3 &= b_{3,0} + b_{3,3}x_3 + b_{3,4}x_1^2 + b_{3,5}x_1x_2.\end{aligned} \tag{12}$$

For this class, $N_m = 9$.

To determine the relations among the parameters $b_{i,*}$, the differential model of system (12) is constructed. The functional form of the differential model is (6) and the relations between the parameters α_i in (6) and the parameters $b_{i,*}$ in Eq. (12) is

$$\begin{aligned}\alpha_1 &= b_{1,2}b_{2,6}b_{3,0} - b_{1,2}b_{2,1}b_{3,3} + b_{1,1}b_{2,2}b_{3,3},\\ \alpha_2 &= b_{1,2}b_{2,6}b_{3,4} - b_{1,1}b_{2,6}b_{3,5},\\ \alpha_3 &= -(b_{1,1} + b_{2,2})b_{3,3},\\ \alpha_4 &= b_{2,6}b_{3,5},\\ \alpha_5 &= -b_{1,1} - b_{2,2},\\ \alpha_6 &= b_{1,1} + b_{2,2} + b_{3,3},\\ \alpha_7 &= 1.\end{aligned} \tag{13}$$

When this set of equations is invertible, a fit using the differential model Eq. (6) can be mapped onto a fit using the model of Eq. (12).

Applying the same scaling argument as for the differential model (7) of the Lorenz system, a set of linear relations for the 14 term scalings in Eq. (13) can be summarized in matrix form,

$$\begin{pmatrix} 0&1&0&0&1&1&0&0&0\\ 0&1&1&0&0&0&1&0&0\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 1&0&0&0&1&0&0&0&1\\ 1&0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&1\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} \log(\lambda_{1,1})\\ \log(\lambda_{1,2})\\ \log(\lambda_{2,1})\\ \log(\lambda_{2,2})\\ \log(\lambda_{2,6})\\ \log(\lambda_{3,0})\\ \log(\lambda_{3,3})\\ \log(\lambda_{3,4})\\ \log(\lambda_{3,5}) \end{pmatrix} = 0. \tag{14}$$

This 14×9 matrix has a two-dimensional null space spanned by the null vectors (0, 1, −1, 0, −1, 0, 0, 0, 1) and (0, −1, 1, 0, 0, 1, 0, 1, 0).

This means that for the general Lorenz-like system (12), there exists a 2-parameter family of equations

$$\begin{aligned}\dot{x}_1 &= b_{1,1}x_1 + \tfrac{1}{n}mb_{1,2}x_2,\\ \dot{x}_2 &= n\tfrac{1}{m}b_{2,1}x_1 + b_{2,2}x_2 + \tfrac{1}{m}b_{2,6}x_1x_3,\\ \dot{x}_3 &= nb_{3,0} + b_{3,3}x_3 + nb_{3,4}x_1^2 + mb_{3,5}x_1x_2.\end{aligned} \tag{15}$$

Once again the scale factor m is constrained by the relation $\frac{1}{m}b_{2,6} + mb_{3,5} = 0$.

The Lorenz system (2) and the general Lorenz-like system (12) will share the same time series $x_1(t)$ only when the set of parameters α_i in Eq. (7)—with $a_{1,1} = -\sigma, a_{1,2} = \sigma, a_{2,1} = R, a_{2,2} = -1, a_{2,6} = -1, a_{3,3} = -b,$ and $a_{3,5} = 1$—and in Eq. (13) are the same

$$\begin{aligned} b\sigma(R-1) &= (b_{1,1}b_{2,2} - b_{1,2}b_{2,1})b_{3,3}\\ &\quad + b_{1,2}b_{2,6}b_{3,0},\\ -\sigma &= -b_{1,1}b_{2,6}b_{3,5} + b_{1,2}b_{2,6}b_{3,4},\\ -b(\sigma+1) &= -(b_{1,1} + b_{2,2})b_{3,3},\\ -1 &= b_{2,6}b_{3,5},\\ 1+\sigma &= -(b_{1,1} + b_{2,2}),\\ -1-b-\sigma &= b_{1,1} + b_{2,2} + b_{3,3},\\ 1 &= 1.\end{aligned} \tag{16}$$

One example for possible choices of the parameters in Eq. (16) is $\sigma = 10, R = 28, b = \frac{8}{3}$ and $b_{1,1} = 1$, $b_{1,2} = 1$, $b_{2,1} = 2061/8, b_{2,2} = -12, b_{2,6} = 1, b_{3,0} = 1, b_{3,3} = -\frac{8}{3}, b_{3,4} = -11, b_{3,5} = -1$, and therefore

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2,\\ \dot{x}_2 &= \tfrac{2061}{8}x_1 - 12x_2 + x_1x_3,\\ \dot{x}_3 &= 1 - \tfrac{8}{3}x_3 - 11x_1^2 - x_1x_2.\end{aligned} \tag{17}$$

This system shares the time series of the x_1-variable with the original Lorenz system (2) for $\sigma = 10, R = 28, b = \frac{8}{3}$.

B. Subsystems of the general Lorenz-like system

There are 17 subsets of the general Lorenz-like system (12) with some of the parameters $b_{i,*}$ set to zero that have the same functional form of the differential model as the original Lorenz system (5). Table I lists all those systems. System (14) is the original Lorenz system (5) and system (1) is the full general Lorenz-like system (12). Only for systems (1)-(12) in Table I, an equation similar to Eq. (16) can be solved. Therefore, only those systems with appropriate parameter choices can share the same time series $x_1(t)$.

This is illustrated with an example.

System 11 is the system published by Wang *et al.*,[4]

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1),\\ \dot{x}_2 &= -x_2 - x_1x_3,\\ \dot{x}_3 &= -R - x_3 + x_1x_2.\end{aligned} \tag{18}$$

For this system, the coefficients of the differential model are

TABLE I. Subsystems of the general Lorenz-like system (12) that have the same functional form of the differential model as the Lorenz system (5). An "x" denotes that the term $b_{i,*}$ is non-zero.

	$b_{1,1}$	$b_{1,2}$	$b_{2,1}$	$b_{2,2}$	$b_{2,6}$	$b_{3,0}$	$b_{3,3}$	$b_{3,4}$	$b_{3,5}$
1	x	x	x	x	x	x	x	x	x
2		x	x	x	x	x	x	x	x
3	x	x		x	x	x	x	x	x
4	x	x	x		x	x	x	x	x
5	x	x	x	x	x		x	x	x
6	x	x	x	x	x	x	x		x
7		x		x	x	x	x	x	x
8		x	x	x	x		x	x	x
9	x	x			x	x	x	x	x
10	x	x		x	x		x	x	x
11	x	x		x	x	x	x		x
12	x	x	x		x		x	x	x
13	x	x	x		x	x	x		x
14	x	x	x	x	x		x		x
15	x	x			x	x	x		x
16	x	x		x	x		x		x
17	x	x	x		x		x		x

$$\begin{aligned}\alpha_1 &= R\sigma - \sigma,\\ \alpha_2 &= -\sigma,\\ \alpha_3 &= -\sigma - 1,\\ \alpha_4 &= -1,\\ \alpha_5 &= \sigma + 1,\\ \alpha_6 &= -\sigma - 2,\\ \alpha_7 &= 1.\end{aligned} \tag{19}$$

The Lorenz system (5) and the Wang system (18) will share the same time series $x_1(t)$ only when the set of parameters α_i in Eqs. (7) and (19) are the same

$$\begin{aligned}R\sigma - \sigma &= a_{1,1}a_{2,2}a_{3,3} - a_{1,2}a_{2,1}a_{3,3},\\ -\sigma &= -a_{1,1}a_{2,6}a_{3,5},\\ -\sigma - 1 &= -a_{1,1}a_{3,3} - a_{2,2}a_{3,3},\\ -1 &= a_{2,6}a_{3,5},\\ \sigma + 1 &= -a_{1,1} - a_{2,2},\\ -\sigma - 2 &= a_{1,1} + a_{2,2} + a_{3,3},\\ 1 &= 1.\end{aligned} \tag{20}$$

Substituting the solutions of Eq. (20), $a_{1,1} = -\sigma, a_{1,2} = \frac{R\sigma}{a_{2,1}}, a_{2,2} = -1, a_{2,6} = -\frac{1}{a_{3,5}}$, and $a_{3,3} = -1$ and the arbitrary choice of the free parameters $a_{2,1} = 1$ and $a_{3,5} = 1$ in the Lorenz system (5) leads to the Lorenz system written in the form

$$\begin{aligned}\dot{x}_1 &= \sigma(Rx_2 - x_1),\\ \dot{x}_2 &= x_1 - x_2 - x_1x_3,\\ \dot{x}_3 &= -x_3 + x_1x_2.\end{aligned} \tag{21}$$

This system generates the same x_1 time series as the Wang system (18).

C. Connection to feedback circuits

Letellier *et al.*[5] used feedback circuits analysis to classify nine Lorenz-like systems. The Ansatz library approach and the feedback circuit analysis yield the same classification of the Lorenz-like systems that are subsystems of the general Lorenz-like system (12). Certain terms in the differential model seem to correspond to certain feedback circuits.

TABLE II. Parameters of some Lorenz-like systems[1,3,4] in Table I.

System	Table I	$a_{1,1}$	$a_{1,2}$	$a_{2,1}$	$a_{2,2}$	$a_{2,6}$	$a_{3,0}$	$a_{3,3}$	$a_{3,4}$	$a_{3,5}$
Lorenz	#14	$-\sigma$	σ	R	-1	-1		$-b$		1
Chen	#14	$-\sigma$	σ	$R-\sigma$	R	-1		$-b$		1
Wang	#11	$-\sigma$	σ		-1	-1	R_a	-1		1

Note that three of the nine Lorenz-like systems in Ref. 5 (Shimizu and Morioka, Rucklidge, and Burke and Shaw) violate the condition that the trilinear form (Lorenz's $a_{ijk}x_ix_jx_k$) vanishes.

Three of the systems[1,3,4] in Ref. 5 have the same differential model as the general Lorenz-like system (12), 4 systems[9–12] have one term less in the differential model, and 2 systems[12,13] have a different differential model and more terms than the general Lorenz-like systems. Here, only the first three systems that have the same differential model as the Lorenz system will be considered. Table II lists these systems as subsystems of the general Lorenz-like system (12). All those systems have the same differential model and the same feedback circuits ($J_{11}J_{22}J_{33}, J_{11}J_{23}J_{32}, J_{33}J_{12}J_{21}$ and $J_{12}J_{23}J_{31}$). Therefore, these systems have similar dynamical behavior. The Lorenz and the Wang systems further can share the same time series as discussed before. A discussion of the changes of the topology of the attractors caused by changes in the feedback circuits can be found in Ref. 5.

IV. SUMMARY AND DISCUSSION

A 3D system can be rewritten as a differential model where the state space variables are one of the coordinates of the original 3D system $X = \phi(x_1(t), x_2(t), x_3(t))$ and its successive derivatives $Y = \dot{X}$ and $Z = \dot{Y}$. The differential model is unique while there exists a class of original 3D systems that share the same differential model. The example of the general Lorenz-like system and a class of Lorenz-like systems is shown.

If the parameters α_i of the differential model are the same for a class of original 3D systems, then this class of systems will share the same time series $X = x(t)$. An example shown here is the Wang system.[4]

The transformation between the 3D systems in the original phase space and the differential model in the differential embedding space can be used to identify relations between the parameters $a_{i,*}$ of the original systems. These relations can be used to explain why some systems with the same functional form can have different dynamics (e.g., the Lorenz and the Chen systems).

The method used in this paper is based on the properties of the algebraic relations between the original 3D system of ODEs and the corresponding differential model. It is shown that a classification of Lorenz-like systems using topological analysis with feedback circuits corresponds to the algebraic classification in this paper.

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